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Regularity Results for Nonlinear Elliptic Systems

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Abstract

We prove partial regularity results for solutions to systems of elliptic partial differential equations with divergence structure, under nonstandard growth conditions. We consider solutions to $-\mathrm{div}\ a(x,u,Du) = b(x,u,Du)$, and use the method of $\mathcal{A}$-harmonic approximation to show $C^{1,\alpha}$ regularity almost-everywhere. We further calculate the optimal exponent, provided the operator a has Hölder continuous coefficients. We then relax the continuity assumption to allow for VMO or small BMO coefficients, and while a loss of regularity in the solution is to be expected, we retain almost-everywhere $C^{0,\alpha}$ regularity in the solution. We then modify a technique of Campanato to further reduce the Hausdorff dimension of the singular set, assuming restrictions on the exponent and ambient dimension.

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Christopher James van der Heide

Publications during candidature

No publications.

Publications included in this thesis

No included publications.

Contributions by others to the thesis

No further contributions by others.

Statement of parts of the thesis submitted to qualify for the award of another degree

None.

In loving memory.

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Elliptic systems, regularity theory, $A-$harmonic approximation, boundary regularity

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Contents

Introduction

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Part I

Preliminary material

We present a brief introduction to regularity theory and partial regularity in the theory of partial differential equations, problems with nonstandard growth, and varying continuity assumptions on the coefficients.

Chapter 1

Background and results

This document assumes a familiarity with the tools necessary for a modern treatment of elliptic partial differential equations (PDE). For an introduction to the analytic aspects of PDE theory, the reader is directed to [Eva11]. For a more comprehensive treatment of elliptic equations in particular, [GT98] is an appropriate resource. Specific definitions of technical terms used in this chapter can be found in Chapter 2.

A brief history

The problem of elliptic regularity stems from attempts to generalise Weyl's lemma, which states that weakly harmonic functions are real analytic [Wey40]. In the linear setting, it was known as early as 1890 that solutions to elliptic equations with analytic coefficients are in fact real analytic [Pic90]. Nonlinear analogues followed as early as 1904 [Ber11], under varying regularity assumptions on the solution (see [Joh55, Mor66]). By the late 1950s, analogous results had been attained for solutions to linear elliptic systems up to the boundary [MN57].

Around the same time, analysis of solutions to nonlinear equations was being conducted, with results from de Giorgi [DG57], Nash [Nas58], and Moser [Mos60] appearing in rapid succession. Each of these authors provided different insights into the lower order regularity of solutions, in the elliptic and parabolic settings. Shortly thereafter, Ladyzhenskaja and Ural'tseva proved that bounded weak solutions are in fact smooth, under relatively mild assumptions [LU61,LU64]. Their results extended up to the boundary, with the notion of boundary regularity of course depending on the regularity of the boundary data.

The success of this programme gave rise to hope that a parallel theory could be developed in the elliptic systems setting. However in 1968 [DG68], de Giorgi provided a counterexample, quashing the notion of a global theory of regularity. He considered the function $u : B^n \to \mathbb{R}^n$ for $n \geq 3$, given by

$$u(x) = \frac{x}{|x|^\sigma}, \quad \text{where} \quad \sigma = \frac{n}{2}\left\{1 - \left[(2n-2)^2 + 1\right]^{-\frac{1}{2}}\right\}$$

which belongs to $W^{1,2}(B^n, \mathbb{R}^n)$. Furthermore, u is a weak solution of the elliptic system

$$\operatorname{div}(A(x) \cdot Du) = 0,$$

where the coefficients are given by

$$A_{ij}^{\alpha\beta}(x) = \delta_{\alpha\beta}\delta_{ij} + \left((n-2)\,\delta_{\alpha i} + n\frac{x_i x_\alpha}{|x|^2}\right)\left((n-2)\,\delta_{\beta j} + n\frac{x_j x_\beta}{|x|^2}\right).$$

Note in particular that u is discontinuous and in fact unbounded at the origin. This system of equations is variational, i.e. it actually arises as the Euler-Lagrange equations to a corresponding energy functional. As such, it additionally provides a counterexample in the variational setting. While de Giorgi's counterexample had discontinuous coefficients, a number of others followed for smoother problems [GM68b, Ne5, vY00, vY02]. With seemingly increasing regularity on the coefficient functions, the nonlinear nature of the systems was in fact masking the discontinuities of the solutions (for discussion see [Min06]).

With hope lost for global regularity, a number of related problems naturally emerged. The focus switched to finding necessary conditions for a solution to be regular at a given point, and attaining some notion of higher order regularity at such points. With the singular set in general being nonempty, upper estimates on its size became important. Provision of sufficient conditions for global regularity were also of interest.

Non-standard growth

Variable exponent spaces were introduced by Orlicz in 1931 [Orl31], when he considered anisotropic ℓ^p spaces and $L^{p(\,\cdot\,)}$ Lebesgue spaces over the real line. Existence and regularity results in PDE theory in such spaces date back to [Mar86] and [Mar87] respectively, where so-called (p, q)-growth conditions were examined. These conditions allow for different exponents in the lower and upper growth bounds (denoted p and q respectively), provided the difference is not too large [Mar91]. Continuous exponent growth conditions embody the borderline case between fixed p and (p, q)-growth conditions, and began appearing in the literature around the same time through the work of Zhikov [Zhi87].

Mathematical models requiring this level of generality encompass those found in the theory of nonlinear elasticity, where the media is allowed to be anisotropic or otherwise highly inhomogeneous, its nonlinear nature varying from point to point in a controlled fashion. Around the turn of the millennium, these problems began to be studied in a more systematic fashion. This was largely due to the work of Rajagopal and Růžička, who discovered their connection to more complex models of non-Newtonian and in particular electrorheological fluids (see [RR96, RR01, R0̊0, R0̊4]).

The prototypical model-cases are the $p(x)$-Laplacian and its non degenerate analogue, which arise as the Euler-Lagrange equations of the variational integrals

$$\mathcal{L}(w,\Omega) := \int_\Omega |Dw|^{p(x)}\,dx, \qquad \text{and} \qquad \mathcal{N}(w,\Omega) := \int_\Omega (1+|Dw|)^{p(x)}\,dx. \tag{1.1}$$

Here we have taken $\Omega \subset\subset \mathbb{R}^n$ to be open, and admit functions $w \in W^{1,p(x)}(\Omega,\mathbb{R}^N)$, a nonstandard Sobolev space (see Chapter 2 for precise definitions). Solutions to these problems and their gradient flows have found applications in image-processing algorithms (see for example [AMS08, CLR06, BCEl$^+$09, LLP10]).

The log-Hölder condition on the modulus of continuity of the exponent $\omega_p(\,\cdot\,)$, given by

$$\limsup_{\rho\downarrow 0} \omega_p(\rho)\log\left(\frac{1}{\rho}\right) \le L \tag{1.2}$$

for some $L > 0$, was shown by Zhikov [Zhi97] to be necessary in order to attain any regularity of solutions. He showed that this is a necessary and sufficient condition for well-posedness of related variational problems [Zhi95]. In [Zhi97], Zhikov provided the first regularity result for solutions to systems with $p(x)$ growth, combining a Caccioppoli inequality with Sobolev-Poincaré and suitable version of Gehring's lemma in order to attain higher integrability. More recent regularity results have considered systems in which the exponent's modulus of continuity is either Hölder continuous or satisfies the vanishing log-Hölder condition, i.e.

$$\limsup_{\rho\downarrow 0} \omega_p(\rho)\log\left(\frac{1}{\rho}\right) = 0. \tag{1.3}$$

Zhikov's estimate is of fundamental importance, allowing localised freezing of the growth exponent. This estimate has proven indispensible in numerous subsequent studies, in effect locally reducing the problem to a fixed exponent one. These techniques have been particularly successful under the further restriction that $p(\,\cdot\,)$ is Hölder continuous. Partial regularity results under this restricted condition can be found for example in [AM01a, CM99] for minimisers of (1.1) and solutions to elliptic systems [AM01b, HZG08] for more general variational integrals, and [Hab08] for minimisers of higher-order functionals. Relaxation of the Hölder continuity assumption to continuity has allowed some 'rougher' regularity results [Hab13]. Furthermore, obstacle problems have been considered [EH10, EH11] and examples of regularity results for stationary electrorheological fluids can be found, for example, in [AM02, BDHS12]. Interpolation estimates for elliptic operators with nonstandard growth can be found in [BH14].

Regularity results in the nonstandard growth setting — as well as the relationship between smoothness assumptions on the coefficients and the regularity of the solution — will be discussed at greater length in the following sections. For a more thorough overview of the variable exponent problems, see the survey article [HHLN10], the monograph [DHHR11], and references therein. Recently more refined estimates have been developed, such as a $p(\cdot)$-Jensen's inequality leading to more refined reverse-Hölder estimates [DS14], and a $p(\cdot)$-Poincaré's estimate [BDB16], which may lead to sharper results.

Analytic techniques

Compactness methods can be used to demonstrate partial regularity results, and fall into two closely related categories. The leitmotif of these methods is that a solution's regularity properties can be deduced by comparison to a related mapping. These maps are taken to be solutions to simplified problems — usually linearisations, mollifications, or 'frozen' versions of the PDE we wish to study.

The first category of techniques is the family of indirect *blow-up methods*, whose key arguments proceed by contradiction. Regularity theorems for elliptic problems were first proved in [GM68a, Mor68], but the techniques themselves date further back to [DG61, Alm68]. Generalisations in a setting closer to the current one are provided by, for instance, [CM99,RT14]. These results merely show the existence of some Hölder exponent $\alpha \in (0,1)$ such that the solution u or its gradient Du belong to the Hölder space $C^{0,\alpha}(\mathrm{Reg}(\Omega))$: they provide no estimate of, or insight into the exponent's actual value. This becomes important when further analysing the singular sets (see [Min08]).

On the other hand, we have the family of direct analytic techniques, comprising of *the direct method* and *harmonic approximation* methods. The techniques employed in this thesis fall into this latter family and will be discussed separately in more detail.

$\mathcal{A}$-harmonic approximation

The theorems contained in Chapter 6 and Chapter 7 are attained via *the method of $\mathcal{A}$-harmonic approximation*. While an argument by contradiction is used in the step showing existence of an appropriate smooth comparison function, the method in itself is still direct enough that we can explicitly calculate the Hölder exponent of the solution u or its gradient Du. This fact becomes crucial when estimating the Hausdorff dimension of the singular set. This is because the current machinery available to deal with systems of this generality requires lower bounds on this Hölder exponent [DKM07,KM06,KM08,KM10]. Furthermore, we are able to keep track of the constants in our estimates, leading to bounds on the Hölder norms and consequently their sensitivity to shifts in structural assumptions on the PDE.

A basic fact from elementary calculus is that local information about a sufficiently regular function is encoded in its linearisation. Similarly, linearisation techniques in PDE theory allow us to deduce local information about solutions to nonlinear PDE, under basic assumptions on the differential equations themselves. Since our various notions of regularity and continuity are local in nature, these linearisation techniques have proven fruitful when applied in this context.

Harmonic approximation techniques once again date back to the work of de Giorgi. First developed as a tool in Geometric Measure Theory, he was able to apply these techniques to prove regularity of minimal surfaces in [DG61]. This work was later extended by Simon in [Sim83], in which he simplified the proof of Allard's regularity theorem [All72] for more general varifolds. A further adaptation by Simon was used to show regularity of energy minimising harmonic maps in [Sim96], again simplifying existing techniques — in this case, the lauded ε-regularity argument of Schoen-Uhlenbeck [SU82].

The basic idea of the method is that if we can control the energy of the solution on a local scale and the solution is 'approximately harmonic' in the sense that we can obtain appropriate bounds when the PDE is integrated against arbitrary test functions, then it must lie close to a harmonic function in the L^2 sense. This fact allows us to compare some 'excess' quantity at different scales, which in turn lets us to deduce partial regularity results. Subsequent generalisations have somewhat complicated this basic premise, whilst always retaining the key ideas.

The method of $\mathcal{A}$-harmonic approximation was developed in [DS02] and [DG00], in the context of minimising variational currents from GMT and nonlinear elliptic systems with quadratic growth in PDE theory, respectively. Instead of a harmonic function, solutions to a more general class of constant coefficient linear elliptic systems can be used as comparison functions. By choice of such a comparison function, the technique can be applied to a wider class of PDE than would otherwise be available. Further extensions followed, obtaining similar results under more general growth assumptions [Bec07,Bec11b] and in a myriad of variational settings [DGG00, DGK04, DGK05, Sch09]. More recently, the use of a Lipschitz-truncation technique has allowed for a direct argument in the existence step, allowing for excision of the contradiction argument, giving a completely direct proof [DLSV12]. We note that use of this technique may allow for use of more convenient excess quantities and may simplify calculations over those found in this thesis.

A note on related techniques — subsequent works have related the solutions to those of the p-Laplacian [DM04] via the *p-harmonic approximation lemma*, and more general nonlinear PDE with polynomial growth in [DM09]. These methods have been particularly successful in resolving questions of partial regularity for degenerate elliptic problems.

A further generalisation, termed the *$\mathcal{AE}$-harmonic approximation lemma*, has been used to attain lower-order regularity on the gradient of solutions to PDE models of stationary electrorheological fluids [BDHS12]. These models assume nonstandard growth and allow for VMO discontinuities in the coefficients, providing a result analogous to [AM02] for VMO systems by use of a harmonic approximation technique. It is feasible that these results may be improved by use of the new $L\log^\gamma L$ estimates found in Chapter 4.

We briefly note some results from the analogous parabolic theory. Duzaar and Mingione introduced the *$\mathcal{A}$-caloric approximation lemma* in [DM05], in which they proved partial regularity of nonlinear parabolic systems with quadratic growth. Systems with superquadratic polynomial growth were subsequently treated in [DMS05], with the subquadratic analogue appearing in [Sch11]. The *p-caloric approximation lemma* was then introduced, and used to attain analogous results for degenerate diffusions in [BDM13]. The Dini continuous analogue appeared in [Bar11] for quadratic systems.

The continuous coefficient case was resolved by Bögelein, Foss, and Mingione in [BFM12] for the case $p \geq 2$, and the subquadratic case was shown in [FG12]. The parabolic analogue of the result from Chapter 6 appeared in [DH12]. This latter work was made possible by higher integrability estimates found in [BD11], which was developed independently from [ZP10], building upon the techniques of [KL00] to provide the parabolic analogue to [Zhi97].

We note it seems unlikely that a hyperbolic analogue will be successfully developed, since in the elliptic and parabolic case the techniques rely upon the smoothness of the comparison maps. However in the hyperbolic case, one does not expect such a priori estimates, even for arbitrary solutions to linear systems.

The direct method and Morrey-type estimates

The main theorem of Chapter 8 is obtained via *the direct method* and *Morrey-type estimates*. We consider systems with nonstandard growth and allow for either *natural* or *controllable growth conditions*, under the very general assumption that the coefficients are continuous. We further impose a restriction on the dimension of the system, roughly speaking, requiring that $p(x) > n - 2$. A partial regularity result is obtained, with the solution u being Hölder continuous for some exponent $\alpha \in (0, 1)$ on the regular set.

The basic outline of the technique is to define a suitable comparison map with favourable decay properties for which good a priori estimates are available, and then in some sense transfer these favourable properties to the solution of interest. In this case the comparison map is taken to be the solution to a homogeneous problem with frozen coefficients and fixed exponent. These estimates are identical to those developed for fixed exponent problems, and have thus been directly retrieved from the works of Arkhipova [Ark03], Beck [Bec08], and Campanato [Cam87a].

The frozen coefficient map then allows us to conduct a comparison argument with our solution u, in turn admitting Morrey-type estimates on the solution's gradient Du. As was the case for the low-order results obtained via harmonic approximation methods, we find that the gradient Du belongs to an appropriate Morrey space, so Hölder continuity of the solution u is then an immediate consequence of the Campanato-Meyer embedding theorem.

The characterisation of the singular set gives an immediate reduction in its Hausdorff dimension (at least on the subset of the domain where the theorem holds). A measure density argument of Giusti [Giu03] lets us estimate the dimension of this set as strictly less than $n - \gamma_1$, where $\gamma_1 > 1$ is the global lower bound on $p(x)$. Since the result holds up to the boundary, we have that almost every boundary point is regular, with respect to the $n-1$ dimensional Hausdorff measure. As mentioned, this result only holds in 'low dimensions' — the case where the ambient dimension is comparable to the nonlinearity. Since the growth exponent varies from point to point, this restriction leads to both local and global versions of the main theorem. The technique also has the advantage that it is adaptable to degenerate and singular systems.

The technique itself dates back to the work of Campanato, who considered systems with simpler structure and controllable growth in the interior [Cam82b, Cam84, Cam87b, Cam87a, Cam88]. These techniques were extended by Arkhipova, Idone, and Beck, who in turn adapted the technique to allow for systems with u dependency and inhomogeneities [Ido04b, Ido04a], up to the boundary in both the superquadratic [Ark97, Ark03] and subquadratic [Bec08] cases. In particular we mention that in the natural growth case, we use the more intricate cutoff procedure developed by Arkhipova in [Ark03] to treat superquadratic systems, later used by Beck in her proof of the subquadratic case [Bec08].

Boundary regularity

Until fifteen years ago, very little was known about about boundary regularity for general elliptic systems with structure similar to those studied in this work, even in the quadratic case. While substantial progress had been made regarding interior regularity, uniqueness results are determined up to some boundary data, and the precise compatibility of solutions with these boundary conditions remained an open problem. Some preliminary results had been obtained, scattered through the literature, pertaining to equations and systems with special structure (see [Col71, HW75, Wie76, JM83]).

Results providing a characterisation of regular boundary points in the general setting were first established by Grotowski in [Gro00], in the fundamental case where $p = 2$ (see [Gro02a, Gro02b]). These were extended to the superquadratic and subquadratic cases in [Ham07] and [Bec07]. The major issue here is that the boundary of an open subset of $\mathbb{R}^n$ has Lebegue measure zero, while primary partial regularity results imply only that the singular set vanishes with respect to this measure. It then becomes imperative to further estimate the Hausdorff dimension of the singular set in order to demonstrate the existence of even a single regular boundary point.

This issue was resolved by Duzaar, Kristensen, and Mingione in [DKM07]. In this work they considered a general class of quadratic systems with controllable growth inhomogeneities, $C^{0,\alpha}$ coefficients in the spatial variables, where the boundary data and domain are of class $C^{1,\alpha}$. They were able to prove that provided the vector field was independent of u, or had $C^{0,\alpha}$ dependence on u in low dimensions, then the Hausdorff dimension of the singular set cannot exceed $n - 2\alpha$. Thus $\mathcal{H}^{n-1}$-almost every boundary point is regular, provided the Hölder exponent $\alpha > \frac{1}{2}$. The analogous result for bounded solutions obeying a natural growth inhomogeneity was then given by Beck in [Bec11a]. Furthermore, in [DKM07] it was shown that the same result can be obtained for homogeneous systems and general $1 < p < \infty$, if the system has no explicit dependence on the solution u, or if the ambient dimension satisfies $p > n-2-\delta$ for some $\delta > 0$. These results extended Mingione's earlier results for the interior, which appeared in [Min03a, Min03b].

The current study makes partial progress extending these results to the nonstandard growth setting. Further extensions are also possible, including deriving the fractional Sobolev-space estimates required for dimension reduction of the singular set, at least in the case where the system has no explicit dependence on the solution u. Progress has been made in this direction, however these results are out of the scope of the current document.

Regularity of coefficients

Hölder continuity results obtained by harmonic approximation methods have the advantage of allowing one to explicitly compute the Hölder exponent appearing in the regularity result. In the case of elliptic systems in divergence form, the gradient Du's Hölder exponent is in some sense inherited from lowest Hölder exponent given in the structure data - that is, modulus of continuity of the system's coefficient functions in the (x, u) variables, the boundary data, and the geometry of the domain. See [DG00, Gro02b, Bec07] for the subquadratic and quadratic cases, and [Ham03, Ham07] for the treatment of superquadratic systems by a related technique. In the variational setting, however, there is a seemingly unavoidable loss of regularity. For a survey of these results see [Min06, DM09] and references therein.

In the nonstandard setting, partial regularity results for variational problems with $p(x)$ growth and Hölder continuous coefficients was proven in [HZG08], although careful tracking of the calculations shows that the estimates on the Hölder exponent are not sharp. The corresponding result for general elliptic systems with nonstandard growth has remained an open problem. The current work provides this result.

Results to date pertaining to gradient continuity for both variational problems and elliptic systems with $p(x)$ growth have assumed Hölder continuity of the exponent function, with lower order regularity requiring the vanishing log-Hölder condition (1.3). Furthermore, where related results have been obtained using harmonic approximation techniques, a loss in regularity of the gradient's Hölder exponent with respect to to the structure data has been observed. In Chapter 6 we are able to close these gaps in the literature, the real novelty appearing in the ability to relax our continuity assumption on the exponent to (1.2), while retaining optimality in the Hölder exponent, and sharpening the characterisation of the singular set in line with the fixed exponent case. The reader should note that relaxing this smoothness restriction on the exponent while retaining Hölder continuity of the coefficients excludes the case of the Euler-Lagrange equations for (1.1).

It has been noted that the gradient's Hölder exponent depends in a precise way upon that of the coefficients. Relaxation of this Hölder continuity assumption on the coefficients therefore results in degeneration of the estimates on gradient's smoothness. In [DG02] it was shown that the so-called Dini continuity condition in some sense plays the borderline role in the limiting case as $\alpha \downarrow 0$. In that work, quadratic systems with Dini continuous coefficients were found to have C^1 solutions, with a priori control on the continuity of the gradient. The subquadratic and superquadratic growth analogues were then proved in [Qui12a] and [Qui12b], respectively. Note that while the abstracts of these papers claim that the Hölder exponent is explicitly calculated, this is not in fact the case.

Pushing past this borderline case to the setting of mere continuity of the coefficients, we can no longer expect gradient continuity. Rather, one would expect certain Morrey space estimates on the gradient, which in turn results in Hölder continuity on the solution itself for any exponent $\alpha \in (0, 1)$. Indeed this was claimed by Campanato in [Cam82b, Cam84, Cam88], however the proof contained an error. While the result held true in the low dimensional case where $n \leq p + 2$ (see [Cam82c, Cam88, Ark03, Bec08]), the more general conjecture remained open. This low dimensional result is extended in Chapter 8 to the so called variable growth setting.

On the other hand, the question of partial regularity of solutions to systems with continuous coefficients in general dimensions was laid to rest by Foss and Mingione [FM08], in the case where $p \geq 2$. Since we are only expecting Hölder continuity of the solution, one would hope to allow for gradient blowup at regular points. Their technique granted this, and enabled them to prove a type of 'ε-regularity' or 'quantisation of singularities' result, whereby in order for a point to be singular (with respect to a fixed Hölder exponent), a certain local energy needs to exceed a finite quantity at all scales. The subquadratic analog can be retrieved from [Hab13], where Habermann treated it as a special case of the nonstandard growth problem under the condition (1.3). In the variable exponent situation, the log-growth condition forces Habermann's characterisation of the singular set to exclude points of gradient blow up. Consequently, this seemingly artificial assumption allows only for a local version of the singularity quantisation which is characteristic of the fixed-exponent, superquadratic case.

It was then observed in [BDHS11] that this continuity assumption on the coefficients can in fact be relaxed in a controlled way, admitting a class of systems that are discontinuous in the spatial variable. In this work, Bögelein, Duzaar, Habermann, and Scheven demonstrated in the superquadratic growth case that provided the discontinuities satisfy the VMO assumption, the gradient belongs to a certain Morrey space. This in turn implies Hölder continuity of the solution u for every exponent $\alpha \in (0, 1)$. This result holds in both the PDE and variational settings. In Chapter 7, we improve their result in a number of ways, covering both the subquadratic and variable exponent cases in one step, while recovering a local version of Foss and Mingione's ε-regularity result. Furthermore, we are able to relax the log-Hölder condition from (1.3) to (1.2), which is new even in the continuous-coefficient case.

We briefly mention some related results. Problems with VMO coefficients have attracted interest in recent years. The well-posedness of such problems dates back to the work of Chiarenza et al., who developed a priori estimates from [CFL91] into an existence theory in [CFL93], see also [BC07]. The theory was further developed by Byun in [Byu05a, Byu05b], see also the references within. The work of Krylov highlights the fact that problems of this type natural arise when studying stochastic processes [KK01, KL04, Kry04]. This subsequently lead to the study of both elliptic and parabolic equations in [Kry07b, Kry07a], as well as problems in control theory [Kry10] and stochastic PDE [Kry09]. Fully nonlinear equations were studied by Dong, Krylov, and Li in [DKL13], but higher dimensional analogues are less well characterised. Higher regularity for viscosity solutions to a class of parabolic equations was recently shown in [Kry14].

Progress has also been made in the elliptic regularity theory. In addition to the aforementioned results in [BDHS11], Ragusa and Tachikawa have attained numerous results for certain functionals in the variational setting [RT05b, RT05a, RT08, RT11] including $p(x)$-harmonic maps with Takabayashi in [RTT13], see [RT14] for review. Calderon-Zygmund type estimates have been deduced in both the elliptic and parabolic cases. Recent estimates of this type for parabolic systems [DMS05, Sch10] require only measurability in the time variable, which coincides with the assumptions for the existence theory for equations laid out in [Kry07a] and a number of subsequent results.

Chapter 2

Setting

Function spaces and notation

Let $n > 2, N \geq 2$, fix $\alpha \in (0,1)$ and $1 < \gamma_1 \leq \gamma_2 < \infty$. Take $\Omega \subset\subset \mathbb{R}^n$ to be an open set with boundary $\partial\Omega$ of class $C^{1,\alpha}$. This ensures that the inward pointing normal vector, $\nu(x_0)$, is well defined at every boundary point x_0. We always assume the growth exponent $p : \Omega \to [\gamma_1, \gamma]$ to be log-Hölder continuous with modulus of continuity $\omega_p : [0,\infty) \to [0,1]$, that is, ω_p satisfies

$$\limsup_{\rho \downarrow 0} \omega_p(\rho) \log\left(\frac{1}{\rho}\right) \leq L. \tag{2.1}$$

In Chapter 8, when dealing with Euler-Lagrange systems, we will strengthen this assumption to vanishing log-Hölder continuity, where we take $L = 0$ so that

$$\lim_{\rho \downarrow 0} \omega_p(\rho) \log\left(\frac{1}{\rho}\right) = 0. \tag{2.2}$$

We note that this is a strictly stronger condition than is needed. In this setting we require only the existence of some small $\delta_p > 0$ such that

$$\limsup_{\rho \downarrow 0} \omega_p(\rho) \log\left(\frac{1}{\rho}\right) \leq \delta_p,$$

however (2.2) is the familiar condition from the literature.

We write $B_\rho(x_0) := \{x \in \mathbb{R}^n \mid |x - x_0| < \rho\}$ for the (open) ball of radius $\rho > 0$ under the usual euclidean metric, centred at the point $x_0 \in \mathbb{R}^n$, and $B_\rho^+(x_0) := \{x \in \mathbb{R}^n \mid x_n > 0, |x - x_0| < \rho\}$ for the upper-half ball centred at $x_0 \in \mathbb{R}^{n-1} \times \{0\}$. When $x_0 \in \mathbb{R}^{n-1} \times \{0\}$, we denote the flat part of the boundary by $\Gamma_\rho(x_0) := \{x \in \mathbb{R}^n \mid x_n = 0, |x - x_0| < \rho\}$. When convenient, we write $B^+ := B_1^+(0)$ and $\Gamma := \Gamma_1(0)$ without any confusion. We denote the intersection of an open ball with the domain by $\Omega_{x_0}^\rho = \Omega \cap B_\rho(x_0)$

Integration is interpreted in the sense of Lebesgue, and we denote by $|A|$ the Lebesgue measure of the measurable set $A \subset \mathbb{R}^n$ with the special case $\alpha_n = |B_1(0)|$. When

$0<|A|<\infty$ and $f\in L^1(A)$ we write $(f)_A$ for the average value of f over A, that is,

$$(f)_A = \fint_A f\,dx = \frac{1}{|A|}\int_A f\,dx.$$

We will also use the more concise notation $(f)_{x,\rho}=(f)_{B(x,\rho)}$ when x is an interior point, with $(f)^+_{x,\rho}=(f)_{B^+(x,\rho)}$ for boundary points.

We will write $\mathcal{H}^d(A)$ for the d-dimensional Hausdorff measure of the set A, and will at times estimate the value of the Hausdorff dimension of A, denoting $\dim_{\mathcal{H}}(A)$ as the supremum over d such that $\mathcal{H}^d(A)=\infty$, which coincides with the infimum over the same argument such that $\mathcal{H}^d(A)=0$.

Constants may and usually will be updated from line to line and will always be chosen greater than unity. Constants with subscripts attached will be fixed. We refer at times to $\mathrm{Bil}(\mathrm{Hom}(\mathbb{R}^n\otimes\mathbb{R}^N))$, the space of bilinear forms on the space of pointwise linear maps $\mathrm{Hom}(\mathbb{R}^n\otimes\mathbb{R}^N)$ from $\mathbb{R}^n$ to $\mathbb{R}^N$.

We consider the spaces $X=X(U;\mathbb{R}^k)$ of functions mapping $U\subset\mathbb{R}^m$ into $\mathbb{R}^k$, and as usual their local variants $X_{loc}(U;\mathbb{R}^k)$ are defined such that $f\in X_{loc}(U;\mathbb{R}^k)$ whenever $X(U';\mathbb{R}^k)$ for every $U'\subset\subset U$. Where it is clear from context we may omit the domain or codomain without causing confusion. In the following, we describe only the spaces of scalar-valued functions, with their vectorial analogues being obvious generalisations.

We begin with the space $C(\Omega)$ of continuous functions over Ω. Closely related are the usual zeroth- and first-order Hölder spaces $C^{0,\alpha}(\overline{\Omega})$ and $C^{1,\alpha}(\overline{\Omega})$ for $\alpha\in(0,1)$. These are Banach spaces, generated by the norms

$$\|u\|_{C^{0,\alpha}} = \|u\|_{C^{0,\alpha}(\overline{\Omega})} := \sup_{x\in\Omega}|u(x)| + [u]_{C^\alpha}$$

and

$$\|u\|_{C^{1,\alpha}} = \|u\|_{C^{1,\alpha}(\overline{\Omega})} := \sup_{x\in\Omega}|u(x)| + \sup_{y\in\Omega}|Du(y)| + [Du]_{C^\alpha},$$

where the C^α seminorm is given by

$$[v]_{C^\alpha} := [v]_{C^\alpha(\overline{\Omega})} := \sup_{x,y\in\overline{\Omega},x\neq y}\left\{\frac{|v(x)-v(y)|}{|x-y|^\alpha}\right\}.$$

For $1\le p<\infty$ we have the usual Lebesgue spaces $L^p(\Omega)$ of all p-integrable Lebesgue measurable functions u, mapping Ω into $\mathbb{R}$. These spaces are generated by the norms

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{p}} < \infty.$$

When $p = \infty$, we take

$$\|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|.$$

The nonstandard analogues of these spaces are Orlicz spaces, which allow for variable exponents. For continuous $p : \Omega \to [\gamma_1, \gamma_2]$, the space $L^{p(\,\cdot\,)}(\Omega)$ consists of the set of $p(\,\cdot\,)$-integrable functions generated by the Luxembourg norm

$$\|u\|_{L^{p(\,\cdot\,)}(\Omega)} := \inf\left\{\lambda : \lambda > 0, \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

The Sobolev space $W^{1,p}(\Omega)$ is the space of all locally integrable functions $u : \Omega \to \mathbb{R}$ such that Du exists in the weak sense and belongs to $L^p(\Omega, \mathbb{R}^n)$, endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega,\mathbb{R}^n)}.$$

Its nonstandard analogue is then the space of locally integrable u with weak derivatives Du in $L^{p(\,\cdot\,)}(\Omega, \mathbb{R}^n)$, equipped with norm

$$\|u\|_{W^{1,p(\,\cdot\,)}(\Omega)} := \|u\|_{L^{p(\,\cdot\,)}(\Omega)} + \|Du\|_{L^{p(\,\cdot\,)}(\Omega,\mathbb{R}^n)}.$$

We denote by $W_0^{1,p}(\Omega, \mathbb{R}^N)$ the closure of compactly-supported smooth functions in $W^{1,p}(\Omega, \mathbb{R}^N)$, and its nonstandard analogue is of course $W_0^{1,p(\,\cdot\,)}(\Omega, \mathbb{R}^N)$.

When dealing with boundary points we will consider a model problem on the unit half-ball, described in Chapter 5. In this setting our solutions belong to the set

$$\|u\|_{W_\Gamma^{1,p(\,\cdot\,)}} := \left\{\|u\|_{W^{1,p(\,\cdot\,)}(B^+)} : u \equiv 0 \;\text{ on }\; \Gamma\right\},$$

with the condition on the right of course being interpreted in the sense of traces.

For background on the standard Sobolev spaces see [Ada75]. A first reference for basic properties of the generalised Sobolev spaces is [KR91], with a more thorough and modern treatment being given in [DHHR11]. It is worth noting that while there are a number of versions of standard inequalities available in these latter spaces, the current study circumvents the need for these, instead relying on local higher integrability estimates. These estimates allow for embeddings into the more familiar fixed exponent spaces, where the standard versions of these inequalities can be applied.

We introduce the Morrey spaces $L^{p,\mu}(\Omega)$, for $1 \le p < \infty$ and $\mu \in [0, n]$ as the space of maps $u \in L^p(\Omega)$ generated by the norm

$$\|u\|_{L^{p,\mu}(\Omega)} := \left(\sup_{\rho>0, x_0\in\Omega} \rho^{-\mu} \int_{\Omega^\rho_{x_0}} |u(x)|^p \, dx\right)^{\frac{1}{p}}. \tag{2.3}$$

This family of norms in some sense measures the concentration of the L^p function u at scale μ. In particular, we note that for $\mu = 0$ we retrieve the usual L^p space, and for $\mu = n$ we retrieve L^∞ by Lebesgue's theorem. When $\mu > n$ the space only contains the trivial function $u \equiv 0$.

Similarly, we have the classical Campanato spaces $\mathcal{L}^{p,\mu}(\Omega)$, for $1 \leq p < \infty$ and $\mu \in [0, n+p]$ as the space of maps $u \in L^p(\Omega)$ with finite seminorm

$$[u]_{\mathcal{L}^{p,\mu}(\Omega)} := \left(\sup_{\rho>0, x_0 \in \Omega} \rho^{-\mu} \int_{\Omega^\rho_{x_0}} |u(x) - (u)_{\Omega^\rho_{x_0}}|^p \, dx \right)^{\frac{1}{p}}.$$

These seminorms measure the oscillation of the L^p function u at scale μ. When $\mu > n+p$, the spaces consist only of constant functions. For all domains considered in this work, there holds $\mathcal{L}^{p,\mu}(\Omega) = L^{p,\mu}(\Omega)$ whenever $\mu \in [0,n)$, and $\mathcal{L}^{p,\mu}(\Omega) = C^{0,\alpha}(\Omega)$ for $\mu \in (n, n+p]$ and $\alpha = \frac{\mu - n}{p}$.

We will use the following isomorphism between Morrey, Campanato, and Hölder spaces. This version is taken from [KM06], summarising results by Campanato (see [Giu03]) and Meyers [Mye64].

Lemma 2.1 (Campanato-Meyers)**.** *Let $B_r \subset \mathbb{R}^n$ be an open ball, $1 < p \leq n$, and $n - p < \lambda \leq n$. If $u \in W^{1,p}(B_r)$ satisfies $Du \in L^{p,\lambda}(B_r, \mathbb{R}^n)$, then $u \in C^{0,\gamma}(\overline{B_r}) \cap \mathcal{L}^{p,\lambda+p}(B_r)$ for $\lambda = 1 - \frac{n-\lambda}{p}$. Furthermore, we have the estimate*

$$[u]_{C^\gamma(\overline{B_r})} \leq c\, [u]_{\mathcal{L}^{p,\lambda+p}(B_r)} \leq c \|Du\|_{L^{p,\lambda}(B_r)}.$$

Here, the constant c depends only on n and p. Furthermore, the same result holds for any Lipschitz domain Ω if we allow the additional dependency of the constant c on the Lipschitz constant of $\partial\Omega$.

Finally, in the case where $\mu = n$ we have $\mathcal{L}^{p,\mu}(\Omega) = \mathcal{L}^{1,\mu}(\Omega) = \mathrm{BMO}(\Omega)$, the space of function with *bounded mean oscillations* on Ω. Closely related is the space $\mathrm{VMO}(\Omega)$, consisting of the subspace of functions in $\mathrm{BMO}(\Omega)$ such that

$$\lim_{r \downarrow 0} \sup_{x_0 \in \Omega, 0 < \rho < r} \fint_{\Omega^\rho_{x_0}} \left| u(x) - (u)_{\Omega^\rho_{x_0}} \right|^p dx = 0.$$

Functions belonging to this space, and in particular those with small BMO seminorms, will play a key role in Chapter 7.

Structure conditions

We consider solutions to a general class of nonlinear, inhomogeneous elliptic systems of second order partial differential equations in divergence form. In particular, we are concerned with weak solutions to the boundary value problem

$$\begin{cases} -\operatorname{div} a(x,u,Du) = b(x,u,Du) & \text{in } \ \Omega \\ \qquad\qquad\qquad u = g & \text{on } \ \partial\Omega, \end{cases} \tag{2.4}$$

for some given boundary data $g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$. As usual, a weak solution is interpreted as any function $u \in W^{1,p(\,\cdot\,)}(\Omega, \mathbb{R}^N)$ satisfying

$$\int_\Omega a(x,u,Du)D\phi\,dx = \int_\Omega b(x,u,Du)\phi\,dx,$$

for all $\phi \in W_0^{1,p(\,\cdot\,)}(\Omega, \mathbb{R}^N)$, with $u\mid_{\partial\Omega}= g$ in the trace sense.

When investigating boundary points, it becomes is to consider the model problem

$$\begin{cases} -\operatorname{div} a(x,u,Du) = b(x,u,Du) & \text{in } \ B^+, \\ \qquad\qquad\qquad u = 0 & \text{on } \ \Gamma. \end{cases} \tag{2.5}$$

In this case, the weak solution $u \in W_\Gamma^{1,p(\,\cdot\,)}(B_\rho^+(x_0), \mathbb{R}^N)$ satisfies

$$\int_{B_\rho^+(x_0)} a(x,u,Du)D\phi\,dx = \int_{B_\rho^+(x_0)} b(x,u,Du)\phi\,dx,$$

for all $\phi \in W_0^{1,p(\,\cdot\,)}(B^+, \mathbb{R}^N)$. We will see in Chapter 5 that with slight abuse of notation, we can equivalently consider the same structural assumptions on a and b.

More precisely, we assume that $a : \Omega \times \mathbb{R}^N \times \operatorname{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$ is a Carathédory vector field, obeying the following structural conditions for fixed $0 < \nu \le L < \infty$, all triples $(x,\xi,z) \in \Omega \times \mathbb{R}^N \times \operatorname{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$, and any $\zeta \in \operatorname{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$:

(A1) Strong uniform ellipticity: $\nu(1+|z|)^{p(x)-2}|\zeta|^2 \le D_z a(x,\xi,z)\zeta\cdot\zeta$,

(A2) Nonstandard $p(x)$ growth: $|a(x,\xi,z)| \le L(1+|z|)^{p(x)-1}$,

(A3) Bounded derivatives in z: $|D_z a(x,\xi,z)| \le L(1+|z|)^{p(x)-2}$,

(A4) Continuous derivatives in z:

$$|D_z a(x,\xi,z) - D_z a(x,\xi,\bar{z})| \le \begin{cases} L\mu\left(\frac{|z-\bar{z}|}{1+|z|+|\bar{z}|}\right)(1+|z|+|\bar{z}|)^{p(x)-2} & 2 \le p(x), \\ L\mu\left(\frac{|z-\bar{z}|}{1+|z|+|\bar{z}|}\right)\left(\frac{1+|z|+|\bar{z}|}{(1+|z|)(1+|\bar{z}|)}\right)^{2-p(x)} & 1 < p(x) < 2. \end{cases}$$

We have taken $\mu : [0,1) \to [0,\infty)$ to be a monotone nondecreasing modulus of continuity, satisfying $\mu(0) = 0$. Without loss of generality we assume μ^2 is concave.

(A5) Hölder continuity in u: $|a(x,\xi,z) - D_z a(x,\hat{\xi},z)| \le L\omega_\xi(|\xi-\hat{\xi}|)(1+|z|)^{p(x)-1}$.

Here, $\omega_\xi : [0,\infty) \to [0,1]$ satisfies $\omega(t) \le \min\{t^\alpha, 1\}$ for some fixed $\alpha \in (0,1)$. We rely upon differing assumptions when comparing values of a at different points in its spatial argument. In Chapter 6 we assume that a is Hölder continuous in its first variable. We note in contrast to the fixed exponent case the assumption of a logarithmic growth term. Even in the model case $a(x,z) = (1+|z|)^{p(x)-1}$ with $p(x)$ smooth, this growth condition arises.

(A6) Continuity in x:

$$|a(x,\xi,z) - a(y,\xi,z)| \le L\omega\left(|x-y|\right)\left[(1+|z|)^{p(x)-1} + (1+|z|)^{p(y)-1}\right] \\ \times \left[1 + \log\left(1+|z|\right)\right].$$

Again, $\omega : [0,\infty) \to [0,1]$ is a modulus of continuity. In Chapter 6 we assume ω satisfies $\omega(t) \le \min\{t^\alpha, 1\}$ for some fixed $\alpha \in (0,1)$, while in Chapter 8 we assume only that ω satisfies the vanishing log-Hölder condition (2.2). On the other hand, in Chapter 7 we do not impose a continuity assumption on a in its first variable. Instead, we require only the VMO condition uniformly in its other variables, allowing for controlled discontinuities.

(A6a) VMO in x:

$$|a(x,\xi,z) - (a(\,\cdot\,,\xi,z))_{\Omega^r_{x_0}}| \le \mathbf{v}_{x_0}(x,r)\left[(1+|z|)^{p(x)-1} + (1+|z|)^{p(x_0)-1}\right] \\ \times \left[1 + \log\left(1+|z|\right)\right],$$

whenever $x \in \Omega^r_{x_0}$, uniformly in ξ and z. Here, $\mathbf{v}_{x_0} : \mathbb{R}^n \times [0,\rho_0] \to [0,2L]$ is a bounded function such that there holds

(VMO) $\lim_{\rho \searrow 0} \mathbf{V}(\rho) = 0, \qquad$ where $\qquad \mathbf{V}(\rho) := \sup_{x_0\in\Omega, r\in(0,\rho]} ⨍_{\Omega^r_{x_0}} \mathbf{v}_{x_0}(x,r)\,dx.$

Finally, we take the inhomogeneous term b to satisfy either a *controllable growth* condition, or that the solution u is bounded under a *natural growth* condition.

(B1) Controllable growth: $b(x,\xi,z) \le L(1+|z|)^{p(x)-1}$.

(B2) Natural growth for bounded solutions: $b(x,\xi,z) \le L_1|z|^{p(x)} + L_2$,

for $L_1, L_2 > 0$ where $L_1 = L_1(\|u\|_{L^\infty})$ satisfies $2L_1\|u\|_{L^\infty} < \nu$. We briefly note that the coercivity condition

(C) Coercivity: $\nu(1+|z|)^{p(x)} - C \le a(x,\xi,z)\cdot z$,

is implied by **(A1)**.

Singular sets

We now define the sets of interest when examining partial regularity. For fixed $u \in L^{p(\cdot)}(\Omega, \mathbb{R}^N)$ we define its regular set as

$$\operatorname{Reg}_u(\Omega) := \{\, x \in \Omega : u \text{ is continuous on some open neighbourhood of } x \,\},$$

and the *singular set* of u as

$$\operatorname{Sing}_u(\Omega) := \Omega \setminus \operatorname{Reg}_u(\Omega).$$

In each of the main results we have different sufficient conditions for a point to belong to these sets, reflecting the different nature of each of the theorems. Although the techniques used differ in detail, they have a common strategy. An excess quantity is first estimated, then shown to either decay of stay bounded at all scales. The assumptions required to obtain these estimates, and the quantities themselves then dictate the structure of the singular sets.

Systems with Hölder continuous coefficients

In Chapter 6 we consider the partial regularity of systems with Hölder continuous coefficients and nonstandard growth conditions. Due to the direct nature of the method of $\mathcal{A}$-harmonic approximation, we are able to attain *optimal* results. Optimality holds in the sense that if the normalised vector field

$$\frac{a(x,u,z)}{(1+|z|)^{p(x)-1}}$$

is Hölder continuous with exponent $\alpha \in (0,1)$ in the x and u variables, then the solution u is of class $C^{1,\alpha}$ off some singular set, with the same Hölder exponent α. Note that in the special case where $p(x) = p$ constant, then we retrieve optimal results analogous to those obtained in the quadratic, superquadratic and subquadratic cases by Duzaar and Grotowski [DG00], Hamburger [Ham07], and Beck [Bec07, Bec08], respectively.

Similar to the results found in these works, we have characterised the singular set of the gradient $\operatorname{Sing}_{Du}(\Omega) \subset (\Sigma_{1,\Omega} \cup \Sigma_{2,\Omega})$, where

$$\Sigma_{1,\Omega} := \left\{ x_0 \in \Omega : \liminf_{\rho \downarrow 0} \fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|\, dx > 0 \right\} \tag{2.6}$$

and

$$\Sigma_{2,\Omega} := \left\{ x_0 \in \Omega : \limsup_{\rho \downarrow 0} \left(|(u)_{x_0,\rho}| + |(Du)_{x_0,\rho}| \right) = \infty \right\}. \tag{2.7}$$

The reader may note that in the subquadratic case we have potentially sacrificed sharpness for clarity in recharacterising these sets compared with those found in [Bec07,CFM98, CM01]. Furthermore, in contrast with previous results (see for example [HZG08]), our characterisation is independent of terms of the form $(|Du|^{p(x)})_{x_0,\rho}$.

When considering boundary points, we again note the works of Grotowski [Gro02b, Gro00], who first used A-harmonic approximation to characterise regular boundary points for general quadratic nonlinear systems in arbitrary dimension. This characterisation was sharpened by Kronz [Kro05] in a variational setting, and Beck [Bec08] in the subquadratic PDE setting. In these works it was observed that for the boundary analogue of $\Sigma_{1,\Omega}$, we need only consider the gradient in the inward direction normal to the boundary.

More precisely, take $D_{\nu(x_0)}u$ to be the gradient of u in the direction of the inward pointing normal vector $\nu(x_0)$. Then $\mathrm{Sing}_{Du}(\partial\Omega) \subset (\Sigma_{1,\partial\Omega} \cup \Sigma_{2,\partial\Omega})$, where

$$\Sigma_{1,\partial\Omega} := \left\{ x_0 \in \partial\Omega : \liminf_{\rho\downarrow 0} \fint_{\Omega\cap B_\rho(x_0)} |D_{\nu(x_0)}(u-g) - (D_{\nu(x_0)}(u-g))_{\Omega\cap B_\rho(x_0)} \otimes \nu(x_0)|\, dx > 0 \right\} \tag{2.8}$$

and

$$\Sigma_{2,\partial\Omega} := \left\{ x_0 \in \partial\Omega : \limsup_{\rho\downarrow 0} \left|(D_{\nu(x_0)}u)_{\Omega\cap B_\rho(x_0)}\right| = \infty \right\}. \tag{2.9}$$

We reiterate that while this set has vanishing n-dimensional Lebesgue measure, the boundary itself has n-dimensional Lebesgue measure zero. As such, it was not until the seminal work of Duzaar, Kristensen, and Mingione [DKM07] that the existence of regular boundary points was shown, even in the quadratic case. This technique was later adapted by Beck for the subquadratic case in low dimensions, and in the quadratic case with natural growth condition [Bec08, Bec11a].

Systems with discontinuous coefficients

The result obtained in Chapter 7 generalises Theorem 1.1 from [FM08] in multiple directions. In that work, Foss and Mingione's considered systems with continuous coefficients and fixed superquadratic growth exponents. The fixed exponent, subquadratic case can be retrieved from [Hab13], where Habermann treated it as a special case of the variable exponent problem. However, Habermann's techniques required him to assume the so-called 'strong logarithmic Hölder continuity condition' (1.3), which we will prove not to be essential.

The distinguishing feature of Theorem 2.3 is that in order for a point to be singular - that is, to fail to be α-Hölder continuous for some fixed $\alpha \in (0,1)$ - we require certain local energy functionals to exceed some energy quanta at all small scales. Although the regularity pertains to the solution u, the functionals considered have dependence on its gradient. In particular, the inhomogeneous nature of the higher integrability estimates in the variable exponent setting means that, in contrast to [FM08], the constants obtained in the Hölder estimates are not uniform in the spatial variable. Instead, they depend in a critical way upon the spatial variable, in addition to the ellipticity and growth bounds of the system, as well as the chosen exponent function p, and of course the desired Hölder exponent $\alpha \in (0,1)$.

As such, we give multiple characterisations of the singular sets. For fixed $\alpha \in (0,1)$, and $u \in W^{1,p(\cdot)}(\Omega, \mathbb{R}^N)$ define its *α-regular set* as

$$\Omega_u^\alpha := \left\{ x \in \Omega : u \in C^{0,\alpha}(N, \mathbb{R}^n) \text{ for some open neighbourhood } N \text{ of } x \right\},$$

and its regular set as

$$\Omega_u := \left\{ x \in \Omega : u \in C^{0,\alpha}(N, \mathbb{R}^n) \text{ for every } \alpha \in (0,1) \right\}.$$

We resolve $\mathrm{Sing}_u \subset (\Omega \setminus \Omega_u^\alpha) \subset \left(\Sigma_{1,\Omega}^\kappa \cup \Sigma_{2,\Omega}^\sigma \cup \Sigma_{3,\Omega}\right)$, where

$$\Sigma_{1,\Omega}^\kappa := \left\{ x_0 \in \Omega : \liminf_{\rho \downarrow 0} \fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}| \, dx \geq \kappa \right\}, \tag{2.10}$$

and

$$\Sigma_{2,\Omega}^\sigma := \left\{ x_0 \in \Omega : \liminf_{\rho \downarrow 0} \rho \fint_{B_\rho(x_0)} |Du| \, dx \geq \sigma \right\}, \tag{2.11}$$

and

$$\Sigma_{3,\Omega} := \left\{ x_0 \in \Omega : \limsup_{\rho \downarrow 0} \left| (Du)_{x_0,\rho} \right| = \infty \right\}, \tag{2.12}$$

for some κ and σ, which we can in principle calculate explicitly, satisfying $\lim_{\alpha \to 1} \kappa, \sigma = 0$. We similarly characterise $(\Omega \setminus \Omega_u) \subset \left(\Sigma_{1,\Omega}^0 \cup \Sigma_{2,\Omega}^0 \cup \Sigma_{3,\Omega}\right)$, where

$$\Sigma_{1,\Omega}^0 := \left\{ x_0 \in \Omega : \liminf_{\rho \downarrow 0} \fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}| \, dx > 0 \right\}, \tag{2.13}$$

and

$$\Sigma_{2,\Omega}^0 := \left\{ x_0 \in \Omega : \liminf_{\rho \downarrow 0} \rho \fint_{B_\rho(x_0)} |Du| \, dx > 0 \right\}. \tag{2.14}$$

Furthermore, we have $\left|\Omega \setminus \Omega_u\right| = \left|\Omega \setminus \Omega_u^\alpha\right| = 0$.

Systems with continuous coefficients in low dimension:

In Chapter 8 we again consider systems whose coefficients are assumed only to be continuous. We adapt the techniques of Campanato [Cam87a] via Arkhipova [Ark03] and Beck [Bec09b], who considered systems with continuous coefficients and fixed growth exponents. We improve upon these results, in the sense that we allow for variable-growth exponents, and only require the minimal continuity condition (2.1) on the exponent function's modulus of continuity. However we are required to impose the stronger condition (2.2) on the coefficient functions, strictly stronger than those found in [Cam87a, Ark03, Bec09b].

In contrast to Chapter 6 and Chapter 7, we adopt a technique based on the direct method and Morrey-type estimates in order to attain the result. This technique has two main limitations. The most obvious of these is the strict relationship between the values the exponent function can take and the dimension of the ambient space. Furthermore, in contrast to Chapter 7, we are unable to obtain Hölder continuity of the solution u for all exponents $\alpha \in (0,1)$. Instead we roughly have that the exponent is bounded above pointwise, by $\min\{\frac{n-2-\varepsilon}{p(x)}, 1\}$, for some $\varepsilon > 0$.

Due to this restriction on the dimension, we mention only the improvements over the previous sections. Note that wherever $p(x)(1+\delta) > n$ we have that the solution u is Hölder continuous via the Morrey-Sobolev embedding theorem (see Lemma 2.1). On the other hand, when $n-2 < p(x)(1+\delta) < n$ we have $\mathrm{Sing}_u(\Omega) \subset \Sigma^\tau_{p,\Omega}$, where

$$\Sigma^\tau_{p,\Omega} := \left\{ x \in B^+ : \liminf_{\rho \downarrow 0} \rho^{p(x)-n} \int_{B^+_\rho(x)} 1 + |Du|^{p(y)}\, dy \geq \tau \right\}.$$

Here $\tau > 0$ is again some constant depending on the structure conditions of the system of PDE, and can in principle be calculated explicitly. We note the similarity between this set and $\Sigma^\sigma_{2,\Omega}$ defined above. In fact, combining Hölder's inequality and Corollary 4.2 allows us to conclude that $\Sigma^0_{2,\Omega} \subset \Sigma^0_{p,\Omega}$, where we have again taken the inequalities in the definitions of $\Sigma^0_{2,\Omega}$ and $\Sigma^0_{p,\Omega}$ to be strict. This result allows us to disregard $\Sigma^\kappa_{1,\Omega}$, albeit at the expense of the sharpness of the Hölder exponent.

However, the main benefit of the approach used in Chapter 8 over the method of $\mathcal{A}$-harmonic approximation is that we obtain immediate estimates on the Hausdorff dimension of the singular set, due to the measure density result from Lemma 3.1, which dates back to Giusti (see [Giu03]). In particular, this allows us to conclude that for systems satisfying $\gamma_1 > n-2-\varepsilon$, there holds

$$\mathcal{H}^{n-\gamma_1}\left(\Sigma^\tau_{p,\Omega}\right) = 0.$$

Main results

The results obtained in Chapter 6 and Chapter 7 allow for the first time the relaxation of the continuity assumption on the exponent function to being log-Hölder continuous, while still retaining Hölder continuity results analogous to the fixed exponent cases.

In the case of Chapter 6, optimal Hölder continuity of the gradient Du is obtained up to the boundary. However, we require Hölder continuity of the coefficients. As such, the result does not capture solutions to the Euler-Lagrange equations for minimisers of (1.1) beyond those of [CM99]. Nevertheless, our results are an improvement over the results in [CM99] even in the more restrictive case, since the techniques used in the current study offer the advantage of direct calculation of the Hölder exponent of the solution's gradient, with no loss in regularity compared to fixed-exponent systems.

In particular, we show the following.

Theorem 2.2. *Let $u \in g + W_0^{1,p(\,\cdot\,)}\big(\Omega, \mathbb{R}^N\big)$ be a weak solution to* (2.4) *under assumptions* **(A1)**–**(A6)***, where the modulus of continuity satisfies $\omega(t) \leq \min\{t^\alpha, 1\}$ for some $0 < \alpha < 1$, and the inhomogeneity b satisfies either* **(B1)** *or* **(B2)***. Then the following hold:*

(i) $\operatorname{Reg}_{Du}(\Omega)$ *is relatively open in* Ω*,*

(ii) $u \in C^{1,\alpha}\big(\operatorname{Reg}_{Du}(\Omega), \mathbb{R}^N\big)$*,*

(iii) $\operatorname{Sing}_{Du}(\Omega) \subset \big(\Sigma_{1,\Omega} \cup \Sigma_{2,\Omega} \cup \Sigma_{1,\partial\Omega} \cup \Sigma_{2,\partial\Omega}\big)$.

Here, α depends only on the structure data. In particular, we have $\mathcal{L}^n\big(\operatorname{Sing}_u(\Omega)\big) = 0$.

This result is optimal in the sense that we cannot expect a solution to have higher regularity than that of the coefficients, even when the coefficients are independent of the solution. This is highlighted by the following example, adapted from [Gro02b] via [Bec09b].

Example 2.2.1. *Fix $n \geq 2$, $n = 1$ and $\beta \in (0,1)$. Let the coefficients $a(x,z)$ for $x, z \in \mathbb{R}^n$ satisfy*

$$a(x,z) = \frac{(1+z^2)^{\frac{p(x)-2}{2}} z}{(1+(1+x_n^\beta)^2)^{\frac{p(x)-2}{2}}(1+x_n^\beta)}.$$

Then $a(x,z)$ satisfies assumptions **(A1)**–**(A6)** *(with $b = 0, \nu = \frac{\gamma_1 - 1}{4}, L = 6, \omega(t) = \min\{1, t^\beta\}$, and arbitrary $p(x)$), where we take the domain Ω to be an appropriate smoothing of B^+. The function*

$$u(x) = x_n + \frac{1}{1+\beta} x_n^{1+\beta}$$

satisfies $-\operatorname{div} a(x,u) = 0$ *and is of class* $C^{1,\beta}$*, but no more regular on* Ω*. Moreover, in relative neighbourhoods of the origin of the form* $\overline{\Omega} \cap \overline{B_\rho^+}$ *for* $0 < \rho < 1$ *we have that* $\partial\Omega \cap \overline{B_\rho^+}$ *is smooth, and* $u(x)$ *is identically zero.*

On the other hand, in Chapter 7 we generalise the result in [Hab13] to systems with VMO coefficients. In a sense we improve upon this result even in the continuous coefficient case, giving a local version of the 'ε-regularity' result from [FM08] for the variable exponent setting. However in contrast to [FM08], our characterisation of singularities does not allow for local blowup of the gradient. Our result can also be viewed as a generalisation of [BDHS11], which treated systems with coefficients in VMO and fixed superquadratic growth exponents. We note that as an analogue of [BDHS11], our techniques could be adapted to treat models of electrorheological fluids, leading to a similar quantisation result.

More precisely, we obtain the following theorem.

Theorem 2.3. *Let* $u \in g + W_0^{1,p(\,\cdot\,)}\big(\Omega, \mathbb{R}^N\big)$ *be a weak solution to* (2.4) *under assumptions* **(A1)**–**(A5)** *and* **(A6a)***, where the inhomogeneity* b *satisfies* **(B1)***. Then there exist* $\kappa, \sigma > 0$ *such that the following hold:*

(i) Ω_u *and* Ω_u^α *are relatively open in* Ω*,*

(ii) $u \in C^{0,\alpha}\big(\Omega_u, \mathbb{R}^N\big)$ *for every* $\alpha \in (0,1)$*, and*
for every fixed $\alpha \in (0,1), u \in C^{0,\alpha}\big(\Omega_u^\alpha, \mathbb{R}^N\big)$

(iii) $\Omega_u \subset \big(\Sigma_{1,\Omega}^0 \cup \Sigma_{2,\Omega}^0 \cup \Sigma_{3,\Omega}\big)$*, and* $\Omega_u^\alpha \subset \big(\Sigma_{1,\Omega}^\kappa \cup \Sigma_{2,\Omega}^\sigma \cup \Sigma_{3,\Omega}\big)$*.*

In particular, we have $\mathcal{L}^n\big(\Omega \setminus \Omega_u\big) = 0$*.*

Finally we consider systems with log-Hölder continuous coefficients in the low-dimensional case in Chapter 8. In order to obtain this result we must restrict to the case where the exponent function satisfies $n - 2 - \varepsilon < p(x) < n + \varepsilon$, where ε is a fixed positive quantity inherited from a higher integrability exponent. This theorem is a variable-exponent analogue of [Ark03,Bec08,Cam87a], and provides a stepping stone towards showing boundary regularity for solution-dependent systems considered as in Chapter 6.

Theorem 2.4. *Fix* $g \in C^1$ *and let* $u \in g + W_\Gamma^{1,p(x)}(\Omega, \mathbb{R}^N)$ *be a weak solution to* (2.4) *under assumptions* **(A1)**–**(A6)***, where the modulus of continuity* ω *satisfies* (2.2)*, and the inhomogeneity* b *satisfies either the controllable growth condition* **(B1)***, or the natural growth condition* **(B2)** *with* $2L_1\|u\|_{L^\infty} < \nu$*. Then there exists an* $\varepsilon_0 > 0$ *such that on the subset* Ω_p *of* Ω *where* $n - 2 - \frac{\varepsilon_0}{2} < p(x)$ *there holds*

$$\dim_{\mathcal{H}}\big(\operatorname{Sing}_u(\Omega_p)\big) < n - \gamma_1.$$

Moreover, we have

$$u \in C^{0,\gamma}_{loc}\big(\operatorname{Reg}_u(\Omega_p)\big),$$

for all $\gamma \in \left(0, \min\left\{1 - \frac{n-2-\frac{\varepsilon_0}{2}}{\gamma_1}, 1\right\}\right)$. *Furthermore, we can characterise the singular set via the enclosure*

$$\operatorname{Sing}_u(\Omega_p) \subset \Sigma_{p,\Omega_p} := \left\{x \in \overline{\Omega} : \liminf_{\rho \downarrow 0} \rho^{p(x)-n} \mathcal{M}_p(x,u,r) > 0\right\}.$$

Here, $\gamma_1 = \inf_{x\in\Omega} p(x)$. Note that for $p(x)(1+\frac{\delta}{2}) > n$ we combine the higher integrability result with the Sobolev-Morrey embedding theorem to attain everywhere Hölder continuity. On the other hand, when $p(x) < n-2-\frac{\varepsilon_0}{2}$ we have almost everywhere Hölder continuity via Theorem 2.3.

Remark 2.5. *In fact, we can provide a local improvement of this characterisation, since by careful tracking through the calculations it is evident that for any point* $x \in \operatorname{Reg}_u(\Omega_p)$ *we have*

$$u \in C^{0,\hat{\gamma}}\big(N\big),$$

for all $\gamma \in \left(0, \min\left\{1 - \frac{n-2-\frac{\varepsilon_0}{2}}{p_M}, 1\right\}\right)$ *and some open neighbourhood* N *of* x. *Here, of course* $p_M = p_M(x) = \sup_{B_{\rho_0}(x)\cap\overline{\Omega}} p(y)$, *where* ρ_0 *is the radius given in Corollary 4.2.*

Chapter 3

Preliminary tools

This chapter presents a collection of tools, results, and estimates. Most of the material will be considered standard by expert readers, and any adaptations to the current setting will be justified with accompanying calculations. The machinery presented will be used in later chapters, and is often shared between them. While the chapters containing the proofs of the main results are largely self contained, we will often refer back to these tools and estimates, effectively making this section a reference for later work.

Decay estimates

We begin with some useful results from geometric measure theory and regularity theory in the analysis of PDE.

When dealing with singular sets it is standard to estimate their Hausdorff dimension by use of a measure density result originally deduced by Giusti [Giu03]. In particular, we have the following:

Lemma 3.1. *Let A be an open subset of $\mathbb{R}^n$ and λ a non-negative and increasing finite set function, defined on the family of open subsets of A that is countably superadditive. That is,*

$$\sum_{i\in\mathbb{N}} \lambda\left(\mathcal{O}_i\right) \le \lambda\left(\bigcup_{i\in\mathbb{N}} \mathcal{O}_i\right)$$

whenever $\{\mathcal{O}_i\}_{i\in\mathbb{N}}$ is a countable family of pairwise disjoint open subsets of A. Then for $0 < \tau < n$ there holds

$$\dim_{\mathcal{H}}(E^\tau) \le \tau,$$

where

$$E^\tau := \left\{ x \in A : \limsup_{\rho\downarrow 0} \rho^{-\tau} \lambda\left(B_\rho(x)\right) > 0 \right\}.$$

From [Gia83] via Lemma 2.2 in [DGK04] we have the following result, which allows us to neglect small but controlled quantities when embedding our solution into a Campanato or Morrey space:

Lemma 3.2. *Let A, B, R_1, α and β be non-negative real numbers with $\alpha > \beta$. Then there exists some positive constant κ_0 and a positive constant c depending only on α, β and A such that the following is true: whenever f is nonnegative and nondecreasing on $(0, R_1)$, and satisfies*

$$f(\rho) \leq \left[A\left(\frac{\rho}{R}\right)^{\alpha} + \kappa\right] f(R) + BR^{\beta},$$

for all $0 < \rho < R$, some $R < R_1$, and some $0 < \kappa < \kappa_0$, then for all $0 < \rho < R$ there holds

$$f(\rho) \leq c\left[\left(\frac{\rho}{R}\right)^{\alpha} f(R) + B\rho^{\beta}\right].$$

Finally we will make use of the following iteration lemma, a standard tool when considering decay estimates. This version is taken from Lemma 7.3 in the classical text [Giu03] via [FM08].

Lemma 3.3 (Iteration Lemma)**.** *Let $f : [0, \rho] \to \mathbb{R}$ be a positive, nondecreasing function satisfying*

$$f(\theta^{k+1}\rho) \leq \theta^{\gamma} f(\theta^{k}\rho) + B(\theta^{k}\rho)^{n}$$

and every $k \in \mathbb{N}$, where $\theta \in (0, 1)$, and $\gamma \in (0, n)$, $B \geq 0$, . Then there exists a constant $c = c(\theta, \gamma, n)$ such that for every $r \in (0, \rho)$ there holds

$$f(r) \leq c\left[\left(\frac{r}{\rho}\right)^{\gamma} f(\rho) + Br^{\gamma}\right].$$

The function V

We will frequently refer to the function $V \equiv V_p : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$V(\xi) := \left(1 + |\xi|^2\right)^{\frac{p-2}{4}} \xi \tag{3.1}$$

for each $\xi \in \mathbb{R}^m$ and fixed $1 < p < \infty$. This function has been used extensively in deriving nonlinear partial regularity results, largely due to its almost-linear behaviour for small ξ and polynomial behaviour for large values of ξ.

In particular, we will call upon the following lemma, which collects a number of useful algebraic properties of V. These properties are by now well known and the versions here are taken from Lemma A.4 in [Hab06]. We remark that all the constants' dependences on the exponents are continuous, and will adopt the permanent convention that unless explicitly stated otherwise, $p = p_2 = \sup_{B_\rho(x_0)} p(\,\cdot\,)$.

Lemma 3.4. *Let $1 < p < \infty$ and $V \equiv V_p : \mathbb{R}^m \to \mathbb{R}^m$ be the function defined in (3.1). Then for any $\xi, \eta \in \mathbb{R}^m$ and $t > 0$ there holds*

(i) $|V(t\xi)| \le \max\{|t|, |t|^{\frac{p}{2}}\}|V(\xi)|$;

(ii) $|V(\xi+\eta)| \le c(p)\left(|V(\xi)| + |V(\eta)|\right)$;

(iii) $c(m,p)|\xi - \eta| \le \frac{|V(\xi)-V(\eta)|}{(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{4}}} \le c(m,p)|\xi-\eta|$;

(iv) $2^{\frac{p-2}{4}}\min\{|\xi|, |\xi|^{\frac{p}{2}}\} \le |V(\xi)| \le \min\{|\xi|, |\xi|^{\frac{p}{2}}\}$ *if* $1 < p < 2$;
$\max\{|\xi|, |\xi|^{\frac{p}{2}}\} \le |V(\xi)| \le 2^{\frac{p-2}{4}}\max\{|\xi|, |\xi|^{\frac{p}{2}}\}$ *if* $2 \le p < \infty$;

(v) $|V(\xi) - V(\eta)| \le c(m,p)|V(\xi-\eta)|$ *for* $1 < p < 2$;
$|V(\xi) - V(\eta)| \le c(m,p,M)|V(\xi-\eta)|$ *for* $2 < p \le \infty$ *and* $|\eta| \le M$;

(vi) $|V(\xi-\eta)| \le c(p,M)|V(\xi) - V(\eta)|$ *for* $|\eta| \le M$.

The following improvement of Lemma 2.2 from [AF89] is taken from Lemma 2.3 in [BDHS12].

Lemma 3.5. *Let $1 < \gamma_1 \le p \le \gamma_2 < \infty$, $\xi, \eta \in \mathbb{R}^m$, and $t > 0$. Then for $c = c(\gamma_1, \gamma_2)$ there holds*

$$c^{-1}(1+|\eta|+|\xi|)^{p-2}|\xi-\eta|^2 \le (1+|\eta|)^{p-2}\left|V\left(\frac{\xi-\eta}{1+|\eta|}\right)\right|^2 \le c(1+|\eta|+|\xi|)^{p-2}|\xi-\eta|^2.$$

We will also at times refer to the 'almost-convexity' of the function V (see Definition 6.1 in [Sch08]). To see this, note that for $z \in \mathbb{R}^k$ and some $c > 1$

$$c^{-1}|V(z)|^2 \le (1+|z|)^{p-2}|z|^2 \le c|V(z)|^2, \tag{3.2}$$

where $z \mapsto (1+|z|)^{p-2}|z|^2$ is a convex map. So allowing for fixed multiplicative constants, we can treat $|V(\,\cdot\,)|^2$ as if it were convex.

We recall the following standard estimates, used when decomposing V. Lemma 3.6 can be retrieved from for example [Cam82a], while Lemma 3.7 is a basic consequence of Lemma 3.4 (*iii*).

Lemma 3.6. *Given $\xi, \eta \in \mathbb{R}^k$ and $q > -1$, there exist constants $c_1(q), c_2(q) > 0$ such that*

$$c_1(q)(1+|\xi|+|\eta|)^q \le \int_0^1 (1+|\xi+t\eta|)^q\,dt \le c_2(1+|\xi|+|\eta|)^q.$$

Lemma 3.7. *For $p > 1$, $\xi, \eta \in \mathbb{R}^k$, and $\varepsilon \in (0,1)$ there holds*

$$(1+|\xi|)^{\frac{p-2}{2}}|\xi||\eta| \leq \varepsilon(1+|\xi|)^{\frac{p-2}{2}}|\xi|^2 + \varepsilon^{1-p}(1+|\eta|)^p, \tag{3.3}$$

and

$$1+|\xi|^p \leq c\Big[(1+|\eta|^p) + (1+|\xi|+|\eta|)^{p-2}|\xi-\eta|^2\Big], \tag{3.4}$$

with the constant c depending only on n, N, and p_2.

We will appeal to the following property of the function V, which states that dimension-wise average values of its arguments are quasi-minimisers in an integral sense. The following is taken from Lemma 6.2 in [Sch08], and is a direct consequence of (3.2).

Lemma 3.8. *Let $p \in [1,\infty)$ and $Du \in L^p(\Omega, \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N))$. Then*

$$\int_\Omega \left|V\left(Du - \Sigma_{k\in I}(D_k u)_\Omega \otimes e_k\right)\right|^2 \, dx \leq c(p) \int_\Omega \left|V\left(Du - \Sigma_{k\in I} A_k \otimes e_k\right)\right|^2 \, dx,$$

for all $I \subset \{1,\ldots,n\}$, and all $A_k \in \mathbb{R}^n$ with $k \in I$.

Affine maps

Fixing $x_0 \in \mathbb{R}^N, \rho > 0$, and take $u \in L^2(B_\rho(x_0), \mathbb{R}^N)$. There exists a unique affine function $\ell_{x_0,\rho} : \mathbb{R}^n \to \mathbb{R}^N$ of the form

$$\ell_{x_0,\rho}(x_0) + D\ell_{x_0,\rho}(x-x_0) \qquad \text{that minimises} \qquad \ell \mapsto \fint_{B_\rho(x_0)} |u-\ell|^2 \, dx.$$

Here, $\ell_{x_0,\rho}(x_0) \in \mathbb{R}^N$ and $D\ell_{x_0,\rho} \in \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$. Indeed, by direct calculation there holds

$$\ell_{x_0,\rho}(x_0) = (u)_{x_0,\rho} \qquad \text{and} \qquad D\ell_{x_0,\rho} = \frac{n+2}{\rho^2} \fint_{B_\rho(x_0)} u \otimes (x-x_0) \, dx. \tag{3.5}$$

An elementary calculation lets us compare affine functions that minimise integrals on concentric balls (see Lemma 2 in [Kro02] for the case $p = 2$, Lemma 2.2 from [Hab13] for general p).

Lemma 3.9. *Fix $p \geq 1$, $\theta \in (0,1)$, and $u \in L^p(B_\rho(x_0), \mathbb{R}^N)$. Denote by $\ell_{x_0,\rho}$ and $\ell_{x_0,\theta\rho}$ the minimising affine functions on balls of radius ρ and $\theta\rho$ of the form (3.5). Then we can estimate*

$$|D\ell_{x_0,\rho} - D\ell_{x_0,\theta\rho}|^p \leq \left(\frac{n+2}{\theta\rho}\right)^p \fint_{B_{\theta\rho}(x_0)} |u - \ell_{x_0,\rho}|^p \, dx. \tag{3.6}$$

More generally, there holds for any affine $\Upsilon : \mathbb{R}^n \to \mathbb{R}^N$

$$|D\ell_{x_0,\rho} - D\Upsilon|^p \leq \left(\frac{n+2}{\rho}\right)^p \fint_{B_\rho(x_0)} |u - \Upsilon|^p \, dx. \tag{3.7}$$

We further have the following quasi-minimisation property of functionals for $\ell_{x_0,\rho}$, taken from Corollary 2.4 in [Hab13].

Lemma 3.10. *Fix* $p \geq 1$, $\lambda > 0$, $u \in L^p(B_\rho(x_0), \mathbb{R}^N)$ *and take* $\ell_{x_0,\rho}$ *defined in* (3.5). *Then for all affine* $\Upsilon : \mathbb{R}^n \to \mathbb{R}^N$ *there holds*

$$\fint_{B_\rho(x_0)} |u - \ell_{x_0,\rho}|^p \, dx \leq c \fint_{B_\rho(x_0)} |u - \Upsilon|^p \, dx,$$

and

$$\fint_{B_\rho(x_0)} |V(\lambda(u - \ell_{x_0,\rho}))|^2 \, dx \leq c \fint_{B_\rho(x_0)} |V(\lambda(u - \Upsilon))|^2 \, dx.$$

Here, the constant c *depends only on* n *and* p*, and the dependence on* p *is continuous.*

A basic corollary of Lemma 3.9 is the following lemma, adapted from [Hab13].

Corollary 3.11. *Fix* $p \geq 1$, $\theta \in (0,1)$ *and* $u \in L^p(B_\rho(x_0), \mathbb{R}^N)$. *Denote by* $\ell_{x_0,\rho}$ *and* $\ell_{x_0,\theta\rho}$ *the minimising affine functions on concentric balls of radius* ρ *and* $\theta\rho$ *of the form* (3.5). *Furthermore, assume the smallness condition*

$$\fint_{B_\rho(x_0)} \left| V\left(\frac{u - \ell_{x_0,\rho}}{\rho(1 + |D\ell_{x_0,\rho}|)}\right)\right|^2 dx \leq \left(\frac{1}{4}\frac{\theta^{n+1}}{n+2}\right)^2 < 1 \tag{3.8}$$

holds. Then we have

$$1 + |D\ell_{x_0,\rho}| \leq 2(1 + |D\ell_{x_0,\theta\rho}|).$$

Proof of Corollary 3.11: We can estimate via (3.7) with $p = 1$

$$\begin{aligned}
1 + |D\ell_{x_0,\rho}| &\leq 1 + |D\ell_{x_0,\theta\rho}| + |D\ell_{x_0,\rho} - D\ell_{x_0,\theta\rho}| \\
&\leq 1 + |D\ell_{x_0,\theta\rho}| + \left(\frac{n+2}{\theta\rho}\right) \fint_{B_{\theta\rho}(x_0)} |u - \ell_{x_0,\rho}| \, dx \\
&\leq 1 + |D\ell_{x_0,\theta\rho}| + \left(\frac{n+2}{\theta^{n+1}}\right) \fint_{B_\rho(x_0)} \left|\frac{u - \ell_{x_0,\rho}}{\rho(1 + |D\ell_{x_0,\rho}|)}\right| dx (1 + |D\ell_{x_0,\rho}|). \qquad (3.9)
\end{aligned}$$

Now, when $p \geq 2$ we can calculate via Hölder's inequality, Lemma 3.4 (*iii*) and the smallness assumption (3.8)

$$
\begin{aligned}
\fint_{B_\rho(x_0)} \left| \frac{u-\ell_{x_0,\rho}}{\rho(1+|D\ell_{x_0,\rho}|)} \right| dx &\le \left(\fint_{B_\rho(x_0)} \left| \frac{u-\ell_{x_0,\rho}}{\rho(1+|D\ell_{x_0,\rho}|)} \right|^2 dx \right)^{\frac{1}{2}} \\
&\le \left(\fint_{B_\rho(x_0)} \left| V\left(\frac{u-\ell_{x_0,\rho}}{\rho(1+|D\ell_{x_0,\rho}|)} \right) \right|^2 dx \right)^{\frac{1}{2}} \\
&\le \frac{1}{2}\left(\frac{\theta^{n+1}}{n+2} \right).
\end{aligned}
$$

Writing $T_+ = \{x \in B_\rho(x_0) : u - \ell_{x_0,\rho} \ge \rho(1+|D\ell_{x_0,\rho}|)\}$ and similarly define the set $T_- = \{x \in B_\rho(x_0) : u - \ell_{x_0,\rho} \le \rho(1+|D\ell_{x_0,\rho}|)\}$, for $1 < p < 2$ we can use the same reasoning to find

$$
\begin{aligned}
&\fint_{B_\rho(x_0)} \left| \frac{u-\ell_{x_0,\rho}}{\rho(1+|D\ell_{x_0,\rho}|)} \right| dx \\
&\quad = \fint_{B_\rho(x_0)} \left| \frac{u-\ell_{x_0,\rho}}{\rho(1+|D\ell_{x_0,\rho}|)} \right| \chi_{T_-}\, dx + \fint_{B_\rho(x_0)} \left| \frac{u-\ell_{x_0,\rho}}{\rho(1+|D\ell_{x_0,\rho}|)} \right| \chi_{T_+}\, dx \\
&\quad \le \left(\fint_{B_\rho(x_0)} \left| \frac{u-\ell_{x_0,\rho}}{\rho(1+|D\ell_{x_0,\rho}|)} \right|^2 \chi_{T_-}\, dx \right)^{\frac{1}{2}} + \left(\fint_{B_\rho(x_0)} \left| \frac{u-\ell_{x_0,\rho}}{\rho(1+|D\ell_{x_0,\rho}|)} \right|^p \chi_{T_+}\, dx \right)^{\frac{1}{q}} \\
&\quad \le \left(\fint_{B_\rho(x_0)} \left| V\left(\frac{u-\ell_{x_0,\rho}}{\rho(1+|D\ell_{x_0,\rho}|)} \right) \right|^2 \chi_{T_-}\, dx \right)^{\frac{1}{2}} + \left(\fint_{B_\rho(x_0)} \left| V\left(\frac{u-\ell_{x_0,\rho}}{\rho(1+|D\ell_{x_0,\rho}|)} \right) \right|^2 \chi_{T_+}\, dx \right)^{\frac{1}{q}} \\
&\quad \le 2\left(\fint_{B_\rho(x_0)} \left| V\left(\frac{u-\ell_{x_0,\rho}}{\rho(1+|D\ell_{x_0,\rho}|)} \right) \right|^2 dx \right)^{\frac{1}{2}} \\
&\quad \le \frac{1}{2}\left(\frac{\theta^{n+1}}{n+2} \right).
\end{aligned}
$$

Plugging these estimates into (3.9) gives

$$
1 + |D\ell_{x_0,\rho}| \le 1 + |D\ell_{x_0,\theta\rho}| + \frac{1}{2}(1+|D\ell_{x_0,\rho}|)
$$

as required. □

Excess functionals

We will ultimately show partial Hölder continuity of solutions to (2.4) via suitable estimates of the Campanato space or Morrey space seminorms. We then use Lemma 2.1 which embeds these spaces into those of Hölder continuous functions. In order to do so we now introduce a number of excess functionals, corresponding to the different quantities we are estimating in each section.

When dealing with interior points in Chapter 6, we make use of the following first order Campanato style excess functional. For every point $x \in \Omega$ with $B_\rho(x) \subset\subset \Omega$, a fixed $u \in W_0^{1,p(\,\cdot\,)}(\Omega, \mathbb{R}^N)$, and $\Lambda \in \mathbb{R}^{nN}$ we define the excess functional by

$$\mathcal{C}(x, \Lambda, \rho) := \fint_{B_\rho(x)} |V(Du - \Lambda)|^2 \, dy, \tag{3.10}$$

with V defined by (3.1), with exponent $p = p_2 = \sup_{B_\rho(x)} p(\,\cdot\,)$.

When investigating boundary points we consider the transformed system described in Chapter 5. In this case, we take the spatial domain as an upper half ball B^+ in $\mathbb{R}^n$. We will use the same notation for functionals in this setting, with the domain of definition being clear from context. For every half ball $B_\rho^+(x)$ with $x \in \Gamma$ and $B_\rho^+(x) \subset\subset B^+$ we fix $u \in W^{1,p(\,\cdot\,)}(B^+, \mathbb{R}^N)$ and $\Lambda \in \mathbb{R}^{nN}$ to define the excess functional

$$\mathcal{C}(x, \Lambda, \rho) := \fint_{B_\rho^+(x)} |V(Du - \Lambda)|^2 \, dy, \tag{3.11}$$

with V again defined by (3.1), where here $p = p_2 = \sup_{B_\rho^+(x)} p(\,\cdot\,)$.

We will shorten this notation to $\mathcal{C}(x, \rho) = \mathcal{C}(x, (Du)_{x,\rho}, \rho)$ for interior points, or $\mathcal{C}(x, (Du)_{x,\rho}^+, \rho)$ for boundary points, with the tacit understanding that the quantity always involves the given weak solution to (2.4).

When investigating systems with less regular coefficients in Chapter 7, we consider a renormalised version of the functional $\mathcal{C}$. For any $x \in \Omega$ with $B_\rho(x) \subset\subset \Omega$, fixed $u \in W^{1,p(\,\cdot\,)}(\Omega, \mathbb{R}^N)$ and $D\ell \in \mathbb{R}^{nN}$ we set

$$\Phi(x, D\ell, \rho) := \fint_{B_\rho(x)} \left| V\left(\frac{Du - D\ell}{1 + |D\ell|} \right) \right|^2 dy. \tag{3.12}$$

We similarly define the normalised zeroth-order excess functional Ψ. For any $x \in \Omega$ with $B_\rho(x) \subset\subset \Omega$, fixed $u \in W^{1,p(x)}(\Omega, \mathbb{R}^N)$ and affine map $\ell : \mathbb{R}^n \to \mathbb{R}^N$, we set

$$\Psi(x, \ell, \rho) := \fint_{B_\rho(x)} \left| V\left(\frac{u - \ell}{\rho(1 + |D\ell|)} \right) \right|^2 dy, \tag{3.13}$$

with V defined by (3.1), again for $p = p_2$.

In contrast to the use of the method of $\mathcal{A}$-harmonic approximation in Chapter 6 and Chapter 7, in Chapter 8 we obtain our partial regularity result via the direct method. To demonstrate this we use a first-order Morrey-type excess functional, which allows us to show Hölder continuity of the solution via Lemma 2.1, and to estimate the Hausdorff dimension of the singular set via Lemma 3.1. For any $x \in B^+$ with $B_r(x) \subset\subset B^+$, and fixed $u \in W^{1,p(\,\cdot\,)}(B^+, \mathbb{R}^N)$ we set

$$\mathcal{M}(x,r) := \int_{B_r(x)} 1 + |Du|^{p_2}\,dy, \quad \text{and} \quad \mathcal{M}_p(x,r) := \int_{B_r(x)} 1 + |Du|^{p(y)}\,dy. \tag{3.14}$$

where of course $p_2 = \sup_{B_r(x)} p(\,\cdot\,)$. Similarly, for boundary points $x \in \Gamma$ with $B_r(x) \subset\subset B$ and fixed $u \in W^{1,p(\,\cdot\,)}(B^+, \mathbb{R}^N)$ consider

$$\mathcal{M}(x,r) := \int_{B_r^+(x)} 1 + |Du|^{p_2}\,dy \quad \text{and} \quad \mathcal{M}(x,r) := \int_{B_r^+(x)} 1 + |Du|^{p(y)}\,dy. \tag{3.15}$$

As usual we will refer to both these versions as the same object, with the domain of definition being clear from context.

The $\mathcal{A}$-harmonic approximation lemma

The proofs of our results in Chapter 6 and Chapter 7 utilise the $\mathcal{A}$-harmonic approximation lemma. First appearing in the literature in [DG00] in the context of regularity theory for partial differential equations, and then in [DS02] in the context of geometric measure theory, the version given here is taken from [Bec11b].

Let $\mathcal{A} : \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N) \times \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N) \to \mathbb{R}$ be an elliptic bilinear form with constant coefficients. We say a function $h \in W^{1,1}(B_\rho(x_0), \mathbb{R}^N)$ or $h \in W^{1,1}(B_\rho^+(x_0), \mathbb{R}^N)$ is $\mathcal{A}$-harmonic if it satisfies

$$\fint_{B_\rho(x_0)} \mathcal{A}(Dh, D\phi)\,dx = 0$$

or in the boundary case

$$\fint_{B_\rho^+(x_0)} \mathcal{A}(Dh, D\phi)\,dx = 0, \qquad h = 0 \text{ on } \Gamma_\rho(0)$$

for all $\phi \in C_0^1(B_\rho^+(x_0), \mathbb{R}^N)$.

Lemma 3.12 ($\mathcal{A}$-harmonic approximation)**.** *Fix $0 < \nu \le L < \infty$, $1 < p_2 < \infty$, and let $\mathcal{A} : \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N) \times \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N) \to \mathbb{R}$ be a bilinear form, which is elliptic in the sense of Legendre-Hadamard with ellipticity constant ν and upper-bound L. Given $\varepsilon > 0$ there exists a $\delta_1 = \delta_1(n, N, p, L/\nu, \varepsilon)$ such that for all $\kappa \in (0,1]$ and every $w \in$*

$W^{1,p_2}(B_\rho(x_0),\mathbb{R}^N)$ *satisfying*

$$\fint_{B_\rho(x_0)} |V(Dw)|^2\,dx \le \kappa^2,$$

$$\left|\fint_{B_\rho(x_0)} \mathcal{A}(Dw,D\phi)\,dx\right| \le \kappa\delta \sup_{B_\rho(x_0)} |D\phi| \qquad \forall \phi \in C_0^1(B_\rho(x_0),\mathbb{R}^N),$$

there exists an $\mathcal{A}$*-harmonic function* $h \in W^{1,p_2}(B_\rho(x_0),\mathbb{R}^N)$ *satisfying*

$$\sup_{B_{\frac{\rho}{2}}(x_0)} \big(|Dh| + \rho|D^2h|\big) \le c_h \quad \textit{and} \quad \fint_{B_{\frac{\rho}{2}}(x_0)} \left|V\left(\frac{w-\kappa h}{\rho}\right)\right|^2 dx \le \kappa^2\varepsilon. \tag{3.16}$$

Here $V = V_{p_2}$, and the constant c_h depends only on n, N, p_2, and the ratio $\frac{L}{\nu}$. The corresponding result for boundary points holds when we replace the domains $B_\rho(x_0)$ by $B_\rho^+(x_0)$.

Chapter 4

Higher integrability

Zhikov's estimate

Partial continuity results such as those found in this document are contingent upon other more basic regularity properties of solutions — in particular the higher integrability of the gradient of weak solutions to (2.4). As such, a starting point for our proof is a higher integrability result for Du. The proof of such a result involves a combination of a Caccioppoli inequality, a Sobolev-Poincaré estimate, and a suitable version of Gehring's lemma. Estimates of this type were first presented by Zhikov in [Zhi97], with a similar result being found for example in [AM05].

The proof of the following lemma requires that the modulus of continuity of p satisfies the log-Hölder condition

$$\limsup_{\rho\downarrow 0} \omega_p(\rho)\log\left(\frac{1}{\rho}\right) \leq L, \tag{4.1}$$

and that for fixed $x_0 \in \Omega$ we first choose some ρ_0 small enough to ensure the inclusion $B_\rho(x_0) \subset \Omega$ holds, with

$$\begin{cases} \omega_p(8n\rho_0) \leq \sqrt{\frac{n+1}{n}} - 1 \\ 0 \leq \omega_p(\rho)\log\left(\frac{1}{\rho}\right) \leq L \quad \text{for all } \rho \leq \rho_0. \end{cases} \tag{4.2}$$

This restriction on ρ_0 will be assumed henceforth without restatement. Under these conditions on ρ_0 and ω, we have the following higher integrability estimate, ultimately due to Zhikov in [Zhi97]. The incarnation we use is a global version, and is taken from §3.1 in [Hab13]. As already mentioned, its proof is quite simple but includes three major tools — a Caccioppoli-type inequality, a Sobolev-Poincaré estimate, and a suitable version of Gehring's lemma.

Lemma 4.1. *Let $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ with $Du \in L^{p(\,\cdot\,)}\big(\Omega, \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)\big)$ be a weak solution to*

$$-\operatorname{div} a(x,u,Du) = b(x,u,Du) \quad \textit{in} \;\; \Omega, \tag{4.3}$$

where a satisfies **(A1)**–**(A3)** *and* **(A5)**, *b satisfies* **(B1)** *or* **(B2)**, *$p : \Omega \to [\gamma_1, \gamma_2]$ for $1 < \gamma_1 \le \gamma_2 < \infty$ satisfies* (4.1), *and the solution u satisfies*

$$\int_\Omega |Du|^{p(x)} \le E < \infty. \tag{4.4}$$

Then there exists an exponent $\hat{\delta} = \hat{\delta}(n, N, \gamma_1, \gamma_2, L/\nu, E)$ such that $|Du|^{p(x)(1+\hat{\delta})} \in L^1_{loc}(\Omega)$. Furthermore, given $\theta \in (0, \frac{1}{2})$ there exists a constant $c_z = c(n, N, \gamma_1, \gamma_2, L/\nu, E, \theta)$ and a radius $\rho_0 = \rho_0(n, N, \gamma_1, \gamma_2, L/\nu, \omega)$ such that for any $\delta \in (0, \hat{\delta}]$, any $0 < \rho \le \rho_0$, and each $x_0 \in \Omega$ satisfying $B_\rho(x_0) \subset\subset \Omega$ there holds

$$\left(\fint_{B_{\theta\rho}(x_0)} |Du|^{p(x)(1+\delta)} \, dx \right)^{\frac{1}{1+\delta}} \le c_z \left(\fint_{B_\rho(x_0)} |Du|^{p(x)} \, dx + 1 \right). \tag{4.5}$$

The dependence on θ is such that the constant blows up as $\theta \to 0$. Note that we could take $\theta \in (0, 1)$ if we allow the constant to blow up as $\theta \to 1$.

Here, we may have relabelled ρ_0 taking the smaller value. If necessary, we may further restrict ρ_0 to satisfy

$$\omega_p(2\rho_0) \le \frac{1}{4}\hat{\delta} \tag{4.6}$$

for all $0 < \rho \le \rho_0$, and define

$$p_1(x_0) := \inf_{B_\rho(x_0)} p(x), \qquad \text{and} \qquad p_2(x_0) := \sup_{B_\rho(x_0)} p(x).$$

While these values of p_1 and p_2 may vary from point to point and for different values of ρ, we will suppress this dependence and consider only a model case. Regarding this latter dependence, we denote

$$p_m(x_0) := \inf_{B_{\rho_0}(x_0)} p(x), \qquad \text{and} \qquad p_M(x_0) := \sup_{B_{\rho_0}(x_0)} p(x),$$

and note that (4.6) allows us to easily calculate

$$p_M - p_m \le p_2 - p_1 \le \omega_p(2\rho_0) \le \frac{1}{4}\hat{\delta}.$$

Hence

$$p_2(1 + \frac{1}{2}\hat{\delta}) \le p_M(1 + \frac{1}{2}\hat{\delta}) \le p_m(1 + \hat{\delta}) \le p_1(1 + \hat{\delta}) \le p(x)(1 + \hat{\delta}), \tag{4.7}$$

for all $x \in B_{\rho_0}(x_0)$ and $0 < \rho \le \rho_0$.

The following refinement of Lemma 4.1 for frozen exponents appears as Remark 3.1

in [Hab13]. Note that the upper bounds on the exponent on the left, together with (4.7) allow us to drop dependence on p_2, since we always consider $\rho < \rho_0$. We remark that the analogous statement holds on half-balls.

Corollary 4.2. *Let $B_{2\rho}(x_0) \subset B_{\rho_0}(x_0) \subset \Omega$ and $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ with $Du \in L^{p(\,\cdot\,)}\big(\Omega, \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)\big)$ satisfying* (4.4) *be a weak solution to* (4.3)*, where a satisfies* **(A1)**–**(A3)** *and* **(A5)**, *b satisfies* **(B1)** *or* **(B2)**, *and $p : \Omega \to \mathbb{R}$ satisfies* (4.1). *Then for any $\theta \in (0,1)$, $p_0 \in [p_1, p_2(1+\frac{\delta}{2})]$ and $p \in [p_1, p_2]$ there holds*

$$\left(\fint_{B_{\theta\rho}(x_0)} |Du|^{p_0}\,dx\right)^{\frac{1}{p_0}} \le c\left(\fint_{B_\rho(x_0)} 1 + |Du|^{p(x)}\,dx\right)^{\frac{1}{p}},$$

where the constant c retains the dependencies of δ. Again, we note that constant to blow up as $\theta \to 1$ or 0.

The following corollary allows us to obtain gradient estimates for our solution in terms of affine functions ℓ, under a certain smallness assumption. This version is immediate from Lemma 3.3 found in [Hab13], with obvious modifications.

Corollary 4.3. *Let $B_{2\rho}(x_0) \subset B_{\rho_0}(x_0) \subset \Omega$, $\ell : \mathbb{R}^n \to \mathbb{R}^N$ be an arbitrary affine function, and let u satisfy the conditions of Lemma 4.1. Then for any $\theta \in (0,2]$ and $p \le p_2$ there holds*

$$(i) \qquad \fint_{B_{\theta\rho}(x_0)} |Du|^p\,dx \le 2^{p_2+1}\,(1+|D\ell|)^p \qquad \textit{whenever} \qquad \Phi(x_0,\theta\rho,\ell) \le 1,$$

and

$$(ii) \qquad \frac{1}{2} \le \frac{1+|(Du)_{\theta\rho,x_0}|}{1+|D\ell|} \le 3 \qquad \textit{whenever} \qquad \Phi(x_0,\theta\rho,\ell) \le \frac{1}{36}.$$

Note that via Lemma 3.4 (i) we always have $\Phi(x_0,\theta\rho,\ell) \le \mathcal{C}(x_0,\theta\rho,\ell)$, and so the smallness condition on Φ can be replaced with one on $\mathcal{C}$.

We will also use the following interpolation estimate for L^p-functions, which allows us to use the log-convexity of L^p-norms to equivalently consider the Lebesgue points of our solution in different L^p-spaces. This version appears as (7.9) in [GT98].

Lemma 4.4. *Let $u \in L^p \cap L^q(\Omega, \mathbb{R}^k)$ for $0 < p < q \le \infty$. Then $u \in L^s(\Omega, \mathbb{R}^k)$ for all $p \le s \le q$, and there holds*

$$\|u\|_{L^s} \le \|u\|_{L^p}^{\theta}\|u\|_{L^q}^{1-\theta},$$

where s satisfies $\frac{1}{s} = \frac{\theta}{p} + \frac{1-\theta}{q}$.

The following estimate is taken from [Hab13], and its proof can be inferred from [AM01a].

Lemma 4.5. *Let $\rho \le \frac{1}{e}$ and $u \in W^{1,1}(B_{2\rho}(x_0), \mathbb{R}^N)$ with $Du \in L^{p(\,\cdot\,)}\big(B_{2\rho}(x_0), \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)\big)$ satisfying* (4.4) *be a weak solution to* (4.3)*, where a satisfies* **(A1)**–**(A3)** *and* **(A5)**, *b satisfies* **(B1)** *or* **(B2)**, *and $p : \Omega \to \mathbb{R}$ satisfies* (4.1)*. Then for any $\gamma > 1$ and $\theta \in (0,1)$ there holds*

$$\int_{B_{\theta\rho}(x_0)} \big(1 + |Du|\big)^{p_2} \big(1 + \log^\gamma \big(1 + |Du|\big)\big)\, dx \le c \log\left(\frac{1}{\rho}\right) \int_{B_{2\rho}(x_0)} \big(1 + |Du|\big)^{p_2}\, dx$$

where the constant c depends only on $n, N, \gamma_1, \gamma_2, L/\nu, \theta$ and E. Again, the constant blows up as $\theta \to 0$, and we may take $\theta \in (0, 2)$ provided we allow the constant to blow up as $\theta \to 2$.

Chapter 5

The boundary case

In Chapter 6 and Chapter 8 we will consider partial regularity *up to the boundary.* Since regularity results are local in nature, when considering interior points of some open set $\Omega \subset\subset \mathbb{R}^n$, we need only consider some small open neighbourhood of a given point, which can always be chosen to be properly contained within Ω. However, uniqueness of a solution is often determined by its interaction with some boundary data function. When analysing the smoothness properties of solutions at points on the boundary $\partial\Omega$, both the geometry of the boundary and the regularity of the prescribed function can potentially play a significant role.

In this chapter we demonstrate that it suffices to consider this boundary set as being the flat part of the boundary of a finite collection of equatorial half balls. Consequently, when considering such points it suffices to consider only points in a model half ball. The system of PDE similarly transforms to one with analogous structural assumptions and vanishing Dirichlet data on these sets.

We first state the structural conditions on both the domain and the PDE that will transform. We then follow [Gro00] and [Bec08] in constructing a diffeomorphism, mapping neighbourhoods of boundary points to equatorial upper half balls. We conclude by demonstrating that the structural assumptions on the system of PDE transform to analogous conditions in our model domain.

Domain and operator structure

Given $\Omega \subset\subset \mathbb{R}^n$, where $\partial\Omega$ is of class $C^{1,\alpha}$, we will be considering the regularity of weak solutions to the boundary value problem

$$\begin{cases} -\operatorname{div} a(x,u,Du) = b(x,u,Du) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

for some given boundary data function $g \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$. See Chapter 2 for the definition of a weak solution.

Given some boundary point $x_0 \in \partial\Omega$, we aim to show that, for the purposes of the current study, it suffices to consider weak solutions to the model problem

$$\begin{cases} -\operatorname{div} \hat{a}(x,u,Du) = \hat{b}(x,u,Du) & \text{in } B^+, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{5.2}$$

We will see in this chapter that the structural assumptions on a and b outlined in Chapter 2 are carried over to $\hat{a}$ and $\hat{b}$.

Diffeomorphism

Taking some point $z \in \partial\Omega$, we note that since Ω is a $C^{1,\sigma}$ domain, the inwards pointing unit normal vector $\nu_{\partial\Omega}(z)$ is well defined. After some affine transformation, we may assume without losing generality that $z = 0$, and $\nu_{\partial\Omega}(z) = -e_n$. Our regularity assumption on Ω further ensures that there exists some $\rho_1 > 0$ and function $f \in C^{1,\sigma}(\mathbb{R}^{n-1})$ satisfying $f(0) = 0, D_i f(0) = 0$ for $i = 1, \dots, n-1$, and

$$\Omega \cap B_{\rho_1}(0) = \left\{ x \in B_{r_1} \mid x_n > f(x') \right\},$$

where $x' = (x_1, \dots, x_{n-1})$.

Define now two maps $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$f_1^i(x) := \begin{cases} x_i & i = 1, \dots, n-1 \\ x_n - f(x') & i = n, \end{cases}$$

and similarly

$$f_2^i(x) := \begin{cases} x_i & i = 1, \dots, n-1 \\ x_n + f(x') & i = n. \end{cases}$$

In particular, f_1 and f_2 are $C^{1,\sigma}$-diffeomorphisms satisfying $f_1 = f_2^{-1}$, with Jacobian matrices given by

$$Df_i(x) = \left(\begin{array}{ccc|c} & & & 0 \\ & \mathrm{Id}_{n-1} & & \vdots \\ & & & 0 \\ \hline (-1)^i D_1 f(x') & \dots & (-1)^i D_{n-1} f(x') & 1 \end{array} \right),$$

and $\det Df_i(x) = 1$, $i = 1, 2$. Note in particular that $x \mapsto f_1(x)$ maps 0 to 0 and $\partial\Omega \cap B_{\rho_1}$

to Γ_{ρ_1}.

Since f is Hölder continuous, we can choose some $0 < \rho_2 \leq \rho_1$ to ensure that $|\nabla f(x')| \leq \frac{1}{2}$, provided $x \in \Omega \cap B_{\rho_2}(x_0)$. Calculations found in, for example Section 2.3 in [Bec05], and Section 3.7 in [Gro00] can be used to determine that for any fixed $w \in \mathbb{R}^n$ there holds

$$\frac{1}{\sqrt{2}}|w| \leq |Df_1(x)w| \leq \sqrt{2}|w|, \tag{5.3}$$

and in particular f is Lipschitz continuous on $\Omega \cap B_{\rho_2}$ with Lipschitz constant bounded between $\frac{1}{\sqrt{2}}$ and $\sqrt{2}$. This gives us the inclusion $B^+_{\frac{\rho}{\sqrt{2}}} \subset f_1(\Omega \cap B_\rho) \subset B^+_{\sqrt{2}\rho}$, and an identical argument and set of calculations for f_2 shows that $\Omega \cap B_{\frac{\rho}{\sqrt{2}}} \subset f_2(B^+_\rho) \subset \Omega \cap B_{\sqrt{2}\rho}$ for each $0 < \rho \leq \rho_2$.

We will further restrict $\rho < \rho_2$ when necessary to ensure that

$$\|f_1\|_{C^{1,\sigma}} \|f_2\|_{C^{1,\sigma}} (2\rho)^\sigma \leq \frac{1}{2\sqrt{2}}. \tag{5.4}$$

Transformed operator

Following [Bec11b], which is in turn based on [Bec05,Bec09a,Gro00], we take our solution u to (5.1) satisfying $u = g$ on $\partial\Omega$, and consider the function

$$v(y) := (\tilde{u} - \tilde{g})(y) = (u - g) \circ f_2(y) = (u - g)(x)$$

for $y \in B^+_\rho$, $x \in f_2(B^+_\rho) \subset \Omega \cap B_{\sqrt{2}\rho}$.

Similarly, for some test function $\varphi \in C^\infty_0\big(\Omega \cap B_{\rho/\sqrt{2}}, \mathbb{R}^N\big)$, we extend φ to be zero outside its domain, we set $\tilde{\varphi}$ to satisfy $\tilde{\varphi}(y) := \varphi\big(f_2(y)\big)$, and remark $\tilde{\varphi} \in C^{1,\alpha}_0\big(B^+_\rho, \mathbb{R}^N\big)$.

Now, since u solves (5.1), keeping in mind the support of φ and the Jacobian determinant of f_2, we compute

$$\begin{aligned}
0 &= \int_\Omega a\big(x, u(x), Du(x)\big) \cdot D\varphi(x)\, dx \\
&= \int_{\Omega \cap B_{\frac{\rho}{\sqrt{2}}}} a\big(x, u(x), Du(x)\big) \cdot D\varphi(x)\, dx \\
&= \int_{B^+_\rho} a\big(f_2(y), u(f_2(y)), Du(f_2(y))\big) \cdot D\varphi(f_2(y))\, dy \\
&= \int_{B^+_\rho} a\big(f_2(y), \tilde{u}(y), D\tilde{u}(y)\,[Df_2(y)]^{-1}\big) \cdot D\tilde{\varphi}(y)[Df_2(y)]^{-1}\, dy \\
&= \int_{B^+_\rho} a\big(f_2(y), (\tilde{v} + \tilde{g})(y), (D\tilde{v} + D\tilde{g})(y)[Df_2(y)]^{-1}\big) \cdot D\tilde{\varphi}(y)[Df_2(y)]^{-1}\, dy.
\end{aligned}$$

Noting $f_1 = f_2^{-1}$ and so $[Df_2(y)]^{-1} = Df_1(f_2(y))$, we can rewrite this as

$$\int_{B_\rho^+} \hat{a}(\,\cdot\,, \tilde{u}, D\tilde{u}) \cdot D\tilde{\varphi}\, dy = 0,$$

where the coefficients $\hat{a}(y, \xi, z)$, for $y \in B_\rho^+, \xi \in \mathbb{R}^N$ and $z \in \mathrm{Hom}\big(\mathbb{R}^n, \mathbb{R}^N\big)$, are given by

$$\hat{a}(y, \xi, p) = a\big(f_2(y), (\xi + \tilde{g})(y), [z + D\tilde{g}(y)]Df_1(f_2(y))\big)Df_1^t(f_2(y)).$$

Here, Df_1^t denotes the matrix transpose of Df_1. In components, this reads

$$\hat{a}_i^\tau(y, \xi, p) = a_i^\kappa\big(f_2(y), (\xi + \tilde{g})(y), [z + D\tilde{g}(y)]Df_1(f_2(y))\big)D_\kappa f_1^\tau(f_2(y)),$$

with the usual convention of summation over repeated indices. When considering $D_z\hat{a}$ as a bilinear form on $\mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$ we find for $\lambda, \overline{\lambda} \in \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$,

$$\begin{aligned}
&D_z\hat{a}_i^\tau(y, \xi, z) \qquad\qquad (5.5)\\
&\quad := D_{j,\mu}\hat{a}_i^\tau(y, \xi, z)\lambda_j^\mu\overline{\lambda}_i^\tau\\
&\quad = D_{j,\mu}\Big[a_i^\kappa\Big(f_2(y), (\xi + \tilde{g})(y), [z + D\tilde{g}(y)]Df_1\big(f_2(y)\big)\Big)\Big]\lambda_j^\mu\overline{\lambda}_i^\tau D_\kappa f_1^\tau\big(f_2(y)\big)\\
&\quad = D_{j,\eta}a_i^\kappa\Big(f_2(y), (\xi + \tilde{g})(y), [z + D\tilde{g}(y)]Df_1\big(f_2(y)\big)\big)D_\eta f_1^\mu\big(f_2(y)\big)\lambda_j^\mu\big[\overline{\lambda}Df_1\big(f_2(y)\big)\big]_i^\kappa\\
&\quad = D_z a_i^\kappa\Big(f_2(y), (\xi + \tilde{g})(y), [z + D\tilde{g}(y)]Df_1\big(f_2(y)\big)\Big)\big[Df_1\big(f_2(y)\big)\lambda\big] \cdot \big[\overline{\lambda}Df_1\big(f_2(y)\big)\big].
\end{aligned}$$

Transformed structure conditions

It remains to check that analogues of **(A1)-(A6)** hold for our operator $\hat{a}$. We begin by defining for some log-Hölder continuous p and $f_2 \in C^{1,\sigma}(B_\rho^+, \Omega \cap B_{\sqrt{2}\rho})$, the composition $\tilde{p}(y) : B_\rho^+ \mapsto \mathbb{R}$, satisfying $\tilde{p}(y) = p(f_2(y))$.

We first check whether $\tilde{p}$ satisfies our log-Hölder condition (4.1). Since p is log-Hölder continuous and f_2 is Lipschitz continuous we have for $x, y \in B_\rho^+$

$$\begin{aligned}
\big|\tilde{p}(y) - \tilde{p}(x)\big| = \big|p\big(f_2(y)\big) - p\big(f_2(x)\big)\big| &\leq \omega_p\big(|f_2(y) - f_2(x)|\big)\\
&\leq \omega_p\big(\sqrt{2}|y - x|\big)\\
&\leq \omega_p\big(2\sqrt{2}\rho\big),
\end{aligned}$$

and so

$$\begin{aligned}
\log\Big(\frac{1}{\rho}\Big)\big|\tilde{p}(y) - \tilde{p}(x)\big| &\leq \log\Big(\frac{1}{\rho}\Big)\omega_p\big(\sqrt{2}|y - x|\big)\\
&= \Big[\log\big(2\sqrt{2}\big) + \log\Big(\frac{1}{2\sqrt{2}\rho}\Big)\Big]\omega_p\big(2\sqrt{2}\rho\big).
\end{aligned}$$

Taking the lim sup as $\rho \downarrow 0$, we recover (4.1).

By further restricting p to be in C^α, it is immediate that for $x, y \in B_\rho^+$ there holds

$$
\begin{aligned}
\big|\tilde{p}(y) - \tilde{p}(x)\big| = \big|p\big(f_2(y)\big) - p\big(f_2(x)\big)\big| &\leq \big[p\big]_\alpha \big|f_2(y) - f_2(x)\big|^\alpha \\
&\leq \|p\|_{C^\alpha} \|Df\|_\infty |y - x|^\alpha,
\end{aligned}
$$

and so $\tilde{p} \in C^\alpha(B_\rho^+)$.

Calculations for growth conditions analogous to **(A1)-(A5)** can be found in [Bec07] for the subquadratic case, and inferred from the quadratic calculations in [Gro00] for the superquadratic case. Since these are all pointwise estimates, the calculations are identical those with fixed exponents. The condition **(A6)** is similar yet distinct from those found in these references, and thus needs separate treatment.

Taking $\hat{y}, \tilde{y} \in B_\rho^+, \xi \in \mathbb{R}^N$ and $z \in \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$ we first define for $t \in [0,1]$

$$
\begin{aligned}
A(t) &:= \big(tDf_1\big(f_2(\tilde{y})\big) + (1-t)Df_1\big(f_2(\hat{y})\big)\big)z \\
B(t) &:= A(t) + tDg\big(f_2(\tilde{y})\big) + (1-t)Dg\big(f_2(\hat{y})\big),
\end{aligned}
$$

and calculate

$$
\begin{aligned}
\Big|\frac{d}{dt}B(t)\Big| &= \Big|\Big(Df_1\big(f_2(\tilde{y})\big) - Df_1\big(f_2(\hat{y})\big)\Big)z + Dg\big(f_2(\tilde{y})\big) - Dg\big(f_2(\hat{y})\big)\Big| \\
&\leq \big|Df_1\big(f_2(\tilde{y})\big) - Df_1\big(f_2(\hat{y})\big)\big||z| + \big|Dg\big(f_2(\tilde{y})\big) - Dg\big(f_2(\hat{y})\big)\big| \\
&\leq |z|\|f_1\|_{C^{1,\sigma}}\big|f_2(\tilde{y}) - f_2(\hat{y})\big|^\sigma + \|Dg\|_\infty\big|f_2(\tilde{y}) - f_2(\hat{y})\big|^\alpha \\
&\leq |z|\|f_1\|_{C^{1,\sigma}}\|f_2\|_{C^{1,\sigma}}|\tilde{y} - \hat{y}|^\sigma + \|Dg\|_\infty\|f_2\|_{C^{1,\sigma}}|\tilde{y} - \hat{y}|^\alpha.
\end{aligned}
$$

Using (5.3) and (5.4) we find

$$
\begin{aligned}
\big|A(t)\big| &= \big|tDf_1\big(f_2(\tilde{y})\big) + (1-t)Df_1\big(f_2(\hat{y})\big)\big||z| \\
&\geq |z|\big(\big|Df_1\big(f_2(\hat{y})\big)\big| - t\big|Df_1\big(f_2(\hat{y})\big) - Df_1\big(f_2(\tilde{y})\big)\big|\big) \\
&\geq |z|\Big(\frac{1}{\sqrt{2}} - t\|f_1\|_{C^{1,\sigma}}\big|f_2(\hat{y}) - f_2(\tilde{y})\big|^\sigma\Big) \\
&\geq |z|\Big(\frac{1}{\sqrt{2}} - \|f_1\|_{C^{1,\sigma}}\|f_2\|_{C^{1,\sigma}}|\hat{y} - \tilde{y}|^\sigma\Big) \\
&\geq \frac{1}{2\sqrt{2}}|z|.
\end{aligned}
$$

Since both

$$
\frac{1}{2\sqrt{2}}|z| \leq |A(t)| \leq |B(t)| + \|Dg\|_\infty,
$$

and

$$|B(t)| \leq |A(t)| + \|Dg\|_\infty \leq \sqrt{2}|z| + \|Dg\|_\infty,$$

we have

$$|z| \leq 2\sqrt{2}\big(|B(t)| + \|Dg\|_\infty\big),$$

and

$$\begin{aligned}
\big(1+|B(t)|^2\big)^{\frac{\tilde{p}(\tilde{y})-1}{2}} &\leq \Big(1+\big(\sqrt{2}|z|+\|Dg\|_\infty\big)^2\Big)^{\frac{\tilde{p}(\tilde{y})-1}{2}} \\
&\leq \big(1+4|z|^2+2\|Dg\|_\infty^2\big)^{\frac{\tilde{p}(\tilde{y})-1}{2}} \\
&\leq 2^{\gamma^2-1}\big(1+\|Dg\|_\infty\big)^{\gamma_2-1}\big(1+|z|^2\big)^{\frac{\tilde{p}(\tilde{y})-1}{2}}.
\end{aligned}$$

We can now calculate

$$\begin{aligned}
&\big|\hat{a}(\tilde{y},\xi,z) - \hat{a}(\hat{y},\xi,z)\big| \\
&\qquad = \Big|\Big(a\Big(f_2(\tilde{y}),(\xi+\tilde{g})(\tilde{y}),\big[z+D\tilde{g}(\tilde{y})]Df_1(f_2(\tilde{y}))\Big)Df_1^t\big(f_2(\tilde{y})\big) \\
&\qquad\qquad - a\Big(f_2(\hat{y}),(\xi+\tilde{g})(\hat{y}),\big[z+D\tilde{g}(\hat{y})]Df_1(f_2(\hat{y}))\Big)\Big)Df_1^t\big(f_2(\hat{y})\big)\Big| \\
&\qquad \leq \Big|a\Big(f_2(\tilde{y}),(\xi+\tilde{g})(\tilde{y}),\big[z+D\tilde{g}(\tilde{y})\big]Df_1(f_2\big(\tilde{y})\big)\Big)\Big(Df_1^t\big(f_2(\tilde{y})\big) - Df_1^t\big(f_2(\hat{y})\big)\Big)\Big| \\
&\qquad\quad + \Big|\Big[a\Big(f_2(\tilde{y}),(\xi+\tilde{g})(\tilde{y}),\big[z+D\tilde{g}(\tilde{y})\big]Df_1\big(f_2(\tilde{y})\big)\Big) \\
&\qquad\qquad - a\Big(f_2(\tilde{y}),(\xi+\tilde{g})(\tilde{y}),\big[z+D\tilde{g}(\tilde{y})\big]Df_1\big(f_2(\hat{y})\big)\Big)\Big]Df_1^t\big(f_2(\hat{y})\big)\Big| \\
&\qquad\quad + \Big|\Big[a\Big(f_2(\tilde{y}),(\xi+\tilde{g})(\tilde{y}),\big[z+D\tilde{g}(\tilde{y})\big]Df_1\big(f_2(\hat{y})\big)\Big) \\
&\qquad\qquad - a\Big(f_2(\tilde{y}),(\xi+\tilde{g})(\hat{y}),\big[z+D\tilde{g}(\hat{y})\big]Df_1\big(f_2(\hat{y})\big)\Big)\Big]Df_1^t\big(f_2(\hat{y})\big)\Big| \\
&\qquad\quad + \Big|\Big[a\Big(f_2(\tilde{y}),(\xi+\tilde{g})(\hat{y}),\big[z+D\tilde{g}(\hat{y})\big]Df_1\big(f_2(\hat{y})\big)\Big) \\
&\qquad\qquad - a\Big(f_2(\hat{y}),(\xi+\tilde{g})(\hat{y}),\big[z+D\tilde{g}(\hat{y})\big]Df_1\big(f_2(\hat{y})\big)\Big)\Big]Df_1^t\big(f_2(\hat{y})\big)\Big| \\
&\qquad = \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV},
\end{aligned}$$

with the obvious notation. We can estimate the first term using **(A2)** to find

$$\begin{aligned}
\mathrm{I} &= \Big|a\Big(f_2(\tilde y),(\xi+\tilde g)(\tilde y),\big[z+D\tilde g(\tilde y)\big]Df_1\big(f_2(\tilde y)\big)\Big)\Big(Df_1^t\big(f_2(\tilde y)\big)-Df_1^t\big(f_2(\hat y)\big)\Big)\Big| \\
&\le \Big|a\Big(f_2(\tilde y),(\xi+\tilde g)(\tilde y),\big[z+D\tilde g(\tilde y)\big]Df_1\big(f_2(\tilde y)\big)\Big)\Big|\Big|\Big(Df_1^t\big(f_2(\tilde y)\big)-Df_1^t\big(f_2(\hat y)\big)\Big)\Big| \\
&\le L\Big(1+\big|\big[z+D\tilde g(\tilde y)\big]Df_1\big(f_2(\tilde y)\big)\big|^2\Big)^{\frac{\tilde p(y)-1}{2}}\|f_1\|_{C^{1,\sigma}}\big|f_2(\tilde y)-f_2(\hat y)\big|^\sigma \\
&\le L\big(1+2|z+D\tilde g(\tilde y)|^2\big)^{\frac{\tilde p(y)-1}{2}}\|f_1\|_{C^{1,\sigma}}\big|f_2(\tilde y)-f_2(\hat y)\big|^\sigma \\
&\le L\big(1+4|z|^2+4|D\tilde g(\tilde y)|^2\big)^{\frac{\tilde p(y)-1}{2}}\|f_1\|_{C^{1,\sigma}}\|f_2\|_{C^{1,\sigma}}|\tilde y-\hat y|^\sigma \\
&\le L\|f_1\|_{C^{1,\sigma}}\|f_2\|_{C^{1,\sigma}}\big(4+4\|D\tilde g\|_\infty^2\big)^{\frac{\gamma_2-1}{2}}\big(1+|z|^2\big)^{\frac{\tilde p(y)-1}{2}}|\tilde y-\hat y|^\sigma.
\end{aligned}$$

For the second term we can use the fundamental theorem of calculus together with **(A3)** to compute

$$\begin{aligned}
\mathrm{II} &\le \Big|\Big[a\Big(f_2(\tilde y),(\xi+\tilde g)(\tilde y),\big[z+D\tilde g(\tilde y)\big]Df_1\big(f_2(\tilde y)\big)\Big) \\
&\qquad - a\Big(f_2(\tilde y),(\xi+\tilde g)(\tilde y),\big[z+D\tilde g(\hat y)\big]Df_1\big(f_2(\hat y)\big)\Big)\Big]Df_1^t\big(f_2(\hat y)\big)\Big| \\
&\le \Big|Df_1^t\big(f_2(\hat y)\big)\Big|\Big|\fint_0^1\frac{d}{dt}\big[a\big(f_2(\tilde y),\xi+\tilde g(\tilde y),B(t)\big)\big]\,dt\Big| \\
&\le \sqrt2\Big|\fint_0^1 D_za\big(f_2(\tilde y),\xi+\tilde g(\tilde y),B(t)\big)\cdot\frac{d}{dt}B(t)\,dt\Big| \\
&\le \sqrt2L\Big|\fint_0^1\big(1+|B(t)|\big)^{\tilde p(\tilde y)-2}\,dt\Big|\big(|z|\|f_1\|_{C^{1,\sigma}}\|f_2\|_{C^{1,\sigma}}|\tilde y-\hat y|^\sigma+\|Dg\|_\infty\|f_2\|_{C^{1,\sigma}}|\tilde y-\hat y|^\alpha\big).
\end{aligned}$$

If $2\le\tilde p(\tilde y)\le\gamma_2<\infty$ we have

$$\begin{aligned}
\mathrm{II} &\le \sqrt2L\|f_2\|_{C^{1,\sigma}}\Big|\fint_0^1\big(1+\|Dg\|_\infty+\sqrt2|z|\big)^{\tilde p(\tilde y)-2}\,dt\Big|\big(|z|\|f_1\|_{C^{1,\sigma}}|\tilde y-\hat y|^\sigma+\|Dg\|_\infty|\tilde y-\hat y|^\alpha\big) \\
&\le \sqrt2L\|f_2\|_{C^{1,\sigma}}\big(\|f_1\|_{C^{1,\sigma}}+\|Dg\|_\infty\big)\big(\sqrt2+\|Dg\|_\infty\big)^{\gamma_2-2}\big(1+|z|\big)^{\tilde p(\tilde y)-1}\big(|\tilde y-\hat y|^\sigma+|\tilde y-\hat y|^\alpha\big).
\end{aligned}$$

On the other hand, for $1<\tilde p(\tilde y)<2$ we have

$$\begin{aligned}
\mathrm{II} &\le \sqrt2L\|f_2\|_{C^{1,\sigma}}\Big|\fint_0^1\big(1+|B(t)|\big)^{\tilde p(\tilde y)-2}\,dt\Big|\Big[\big(2\sqrt2|B(t)|+\|Dg\|_\infty\big)\|f_1\|_{C^{1,\sigma}}|\tilde y-\hat y|^\sigma \\
&\qquad + (1+|B(t)|)\|Dg\|_\infty|\tilde y-\hat y|^\alpha\Big] \\
&\le 4L\big(1+\|Dg\|_\infty\big)\fint_0^1\big(1+|B(t)|\big)^{\tilde p(\tilde y)-1}\,dt\big(\|f_1\|_{C^{1,\sigma}}+\|Dg\|_\infty\big)\big(|\tilde y-\hat y|^\sigma+|\tilde y-\hat y|^\alpha\big) \\
&\le 4L\big(1+\|Dg\|_\infty\big)\big(\|f_1\|_{C^{1,\sigma}}+\|Dg\|_\infty\big)\big(1+\sqrt2|z|+\|Dg\|_\infty\big)^{\tilde p(\tilde y)-1}\big(|\tilde y-\hat y|^\sigma+|\tilde y-\hat y|^\alpha\big) \\
&\le 4\sqrt2L\big(1+\|Dg\|_\infty\big)^{\gamma_2}\big(\|f_1\|_{C^{1,\sigma}}+\|Dg\|_\infty\big)\big(1+|z|\big)^{\tilde p(\tilde y)-1}\big(|\tilde y-\hat y|^\sigma+|\tilde y-\hat y|^\alpha\big).
\end{aligned}$$

The third term becomes via **(A5)**

$$
\begin{aligned}
\mathrm{III} &= \Big|\Big[a\Big(f_2(\tilde y), (\xi+\tilde g)(\tilde y), \big[z + D\tilde g(\hat y)\big] Df_1\big(f_2(\hat y)\big)\Big) \\
&\qquad - a\Big(f_2(\tilde y), (\xi+\tilde g)(\hat y), \big[z + D\tilde g(\hat y)\big] Df_1\big(f_2(\hat y)\big)\Big)\Big] Df_1^t\big(f_2(\hat y)\big)\Big| \\
&\le \sqrt{2} L \omega_\xi\big(\big|\tilde g(\tilde y) - \tilde g(\hat y)\big|\big)\Big(1 + \big|\big[z + D\tilde g(\hat y)\big] Df_1\big(f_2(\hat y)\big)\big|^2\Big)^{\frac{\tilde p(\tilde y)-1}{2}} \\
&\le \sqrt{2} L \big(4 + 4\|D\tilde g\|_\infty^2\big)^{\frac{\gamma_2-1}{2}} \omega_\xi\big(\big|\tilde g(\tilde y) - \tilde g(\hat y)\big|\big)\big(1+|z|^2\big)^{\frac{\tilde p(\tilde y)-1}{2}} \\
&\le \sqrt{2} L \big(4 + 4\|D\tilde g\|_\infty^2\big)^{\frac{\gamma_2-1}{2}} \omega_\xi\big(\|g\|_{C^{1,\eta}}\big|f_2(\tilde y) - f_2(\hat y)\big|\big)\big(1+|z|^2\big)^{\frac{\tilde p(\tilde y)-1}{2}} \\
&\le \sqrt{2} L \big(4 + 4\|D\tilde g\|_\infty^2\big)^{\frac{\gamma_2-1}{2}} \omega_\xi\big(\|g\|_{C^{1,\eta}}\|f_2\|_{C^{1,\eta}}|\tilde y - \hat y|\big)\,\big(1+|z|^2\big)^{\frac{\tilde p(\tilde y)-1}{2}} \\
&\le \sqrt{2} L \big(4 + 4\|D\tilde g\|_\infty^2\big)^{\frac{\gamma_2-1}{2}} \|g\|_{C^{1,\eta}}^{\alpha_1}\|f_2\|_{C^{1,\eta}}^{\alpha_1}\omega_\xi\big(|\tilde y - \hat y|\big)\big(1+|z|^2\big)^{\frac{\tilde p(\tilde y)-1}{2}}.
\end{aligned}
$$

In estimating the final term, we write $\tilde z = \big[z + D\tilde g(\hat y)\big] Df_1\big(f_2(\hat y)\big)$ and estimate via **(A6)**

$$
\begin{aligned}
\mathrm{IV} &= \big|\big[a\big(f_2(\tilde y), (\xi+\tilde g)(\hat y), \tilde z\big) - a\big(f_2(\hat y), (\xi+\tilde g)(\hat y), \tilde z\big)\big] Df_1^t\big(f_2(\hat y)\big)\big| \\
&\le \big|a\big(f_2(\tilde y), (\xi+\tilde g)(\hat y), \tilde z\big) - a\big(f_2(\hat y), (\xi+\tilde g)(\hat y), \tilde z\big)\big|\big|Df_1^t\big(f_2(\hat y)\big)\big| \\
&\le \sqrt{2} L \omega\big(\big|f_2(\tilde y) - f_2(\hat y)\big|\big)\Big[\big(1+|\tilde z|^2\big)^{\frac{\tilde p(\tilde y)-1}{2}} + \big(1+|\tilde z|^2\big)^{\frac{\tilde p(y)-1}{2}}\Big]\big[1 + \log\big(1+|\tilde z|^2\big)\big].
\end{aligned}
$$

We have already seen

$$
\big(1+|\tilde z|^2\big)^{\frac{\tilde p(\tilde y)-1}{2}} \le \big(4 + 4\|D\tilde g\|_\infty^2\big)^{\frac{\gamma_2-1}{2}}\big(1+|z|^2\big)^{\frac{\tilde p(\tilde y)-1}{2}},
$$

and similarly

$$
\begin{aligned}
\log\big(1+|\tilde z|^2\big) &= \log\big(1 + |[z + D\tilde g(\hat y)] Df_1(f_2(\hat y))|^2\big) \\
&\le \log\big(1 + 2|z + D\tilde g(\hat y)|^2\big) \\
&\le \log\big(1 + 4|z|^2 + 4|D\tilde g(\hat y)|^2\big) \\
&\le \log\Big(\big(4 + 4|D\tilde g(\hat y)|^2\big)\big(1+|z|^2\big)\Big) \\
&\le \log\big(4 + 4|D\tilde g(\hat y)|^2\big) + \log(1+|z|^2) \\
&\le 4 + 4|D\tilde g(\hat y)|^2 + \log\big(1+|z|^2\big) \\
&\le 4|D\tilde g(\hat y)|^2\big[1 + \log\big(1+|z|^2\big)\big].
\end{aligned}
$$

We can easily calculate

$$
\omega\big(\big|f_2(\tilde y) - f_2(\hat y)\big|\big) \le \omega\big(\|f_2\|_{C^{1,\sigma}}|\tilde y - \hat y|\big),
$$

which is again continuous in its argument. Clarifying, in the case where $\omega(\rho) \le \rho^\alpha$ for some $\alpha \in (0,1)$ we can improve this estimate to

$$\omega\big(\big|f_2(\tilde{y}) - f_2(\hat{y})\big|\big) \le \|f_2\|_{C^{1,\sigma}}^{\alpha}\omega\big(|\tilde{y} - \hat{y}|\big).$$

Compiling these terms we find

$$\begin{aligned}
\mathrm{IV} &= \big|\big[a\big(f_2(\tilde{y}), (\xi + \tilde{g})(\tilde{y}), \tilde{z}\big) - a\big(f_2(\hat{y}), (\xi + \tilde{g})(\hat{y}), \tilde{z}\big)\big] Df_1^t\big(f_2(\hat{y})\big)\big| \\
&\le \sqrt{2}L\big(4 + 4\|D\tilde{g}\|_\infty^2\big)^{\frac{\gamma_2+1}{2}}\omega\big(\|f_2\|_{C^{1,\sigma}}|\tilde{y} - \hat{y}|\big) \\
&\qquad \times \left[\big(1 + |z|^2\big)^{\frac{\tilde{p}(\tilde{y})-1}{2}} + \big(1 + |z|^2\big)^{\frac{\tilde{p}(y)-1}{2}}\right]\big[1 + \log(1 + |z|^2)\big].
\end{aligned}$$

Collecting our estimates for I-IV, we obtain

$$\begin{aligned}
|\hat{a}(\tilde{y}, \xi, z) - \hat{a}(y, \xi, z)| \le \tilde{L}\hat{\omega}\big(|y - \tilde{y}|\big)&\left[\big(1 + |z|^2\big)^{\frac{p(x)-1}{2}} + \big(1 + |z|^2\big)^{\frac{p(\bar{x})-1}{2}}\right] \\
&\times \left[1 + \log\left(1 + |z|^2\right)\right],
\end{aligned}$$

where $\hat{\omega}(\,\cdot\,) = \max\{\omega(\,\cdot\,), \omega_\xi(\,\cdot\,), |\cdot|^{\alpha_1}, |\cdot|^{\sigma}\}$ and the constant $\tilde{L}$ depends on L, f_1, f_2, g and γ_2.

Since these conditions are all preserved under the boundary transformations, in the sequel we will refer to the transformed operator as a without confusion.

Part II

Partial regularity results

We prove new partial regularity results for nonlinear elliptic systems, including almost-everywhere $C^{1,\alpha}$ boundary regularity for systems with Hölder coefficients, almost-everywhere $C^{0,\alpha}$ boundary regularity for systems with VMO-coefficients, and singular set dimension reduction for systems with continuous coefficients in low dimensions.

Chapter 6

Systems with Hölder continuous coefficients

In this chapter, we present the proof of Theorem 2.2. We consider the partial regularity of weak solutions to the inhomogeneous systems of nonlinear elliptic PDE in divergence form of the type

$$\begin{cases} -\operatorname{div} a(x,u,Du) = b(x,u,Du) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

for some given boundary data function $g \in C^{1,\alpha}(\overline{\Omega},\mathbb{R}^N)$. Here a weak solution is interpreted, in the usual sense, as any function $u \in W^{1,p(\,\cdot\,)}(\Omega,\mathbb{R}^N)$ satisfying

$$\int_\Omega a(x,u,Du)D\phi\,dx = \int_\Omega b(x,u,Du)\phi\,dx$$

for all fixed $\phi \in W_0^{1,p(\,\cdot\,)}(\Omega,\mathbb{R}^N)$, where $u\,|_{\partial\Omega}= g$ in the trace sense.

As outlined in Chapter 5, when dealing with questions of regularity at the boundary, it is convenient to consider the transformed operator on the model problem

$$\begin{cases} -\operatorname{div} \hat{a}(x,u,Du) = \hat{b}(x,u,Du) & \text{in } B^+, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{6.2}$$

where our weak solution is now interpreted as a function $u \in W_\Gamma^{1,p(\,\cdot\,)}(B^+,\mathbb{R}^N)$ that satisfies

$$\int_{B^+} a(x,u,Du)D\phi\,dx = \int_{B^+} b(x,u,Du)\phi\,dx$$

for any given test function $\phi \in W_0^{1,p(\,\cdot\,)}(B^+,\mathbb{R}^N)$. In practice a and $\hat{a}$, and b and $\hat{b}$ differ only in the size of the ellipticity and growth bounds, and the moduli of continuity. For ease of notation we will omit the carets.

The operator $a : \Omega \times \mathbb{R}^N \times \operatorname{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$ is a Carathédory vector field, satisfies the follwing assumptions. For fixed $0 < \nu \le L < \infty$, all triples $(x,\xi,z) \in \Omega \times \mathbb{R}^N \times \operatorname{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$, and any $\zeta \in \operatorname{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$:

(H1) Strong uniform ellipticity: $\nu(1+|z|)^{p(x)-2}|\zeta|^2 \le D_z a(x,\xi,z)\zeta\cdot\zeta$,

(H2) Nonstandard $p(x)$ growth: $|a(x,\xi,z)| \le L(1+|z|)^{p(x)-1}$,

(H3) Bounded derivatives in z: $|D_z a(x,\xi,z)| \le L(1+|z|)^{p(x)-2}$,

(H4) Continuous derivatives in z:

$$|D_z a(x,\xi,z) - D_z a(x,\xi,\bar{z})| \le \begin{cases} L\mu\left(\frac{|z-\bar{z}|}{1+|z|+|\bar{z}|}\right)(1+|z|+|\bar{z}|)^{p(x)-2} & 2\le p(x), \\ L\mu\left(\frac{|z-\bar{z}|}{1+|z|+|\bar{z}|}\right)\left(\frac{1+|z|+|\bar{z}|}{(1+|z|)(1+|\bar{z}|)}\right)^{2-p(x)} & 1<p(x)<2, \end{cases}$$

(H5) Hölder continuity in u: $|a(x,\xi,z) - D_z a(x,\hat{\xi},z)| \le L\omega_\xi(|\xi-\hat{\xi}|)(1+|z|)^{p(x)-1}$,

(H6) Hölder continuity in x:

$$\begin{aligned} |a(x,\xi,z) - a(y,\xi,z)| \le L\omega\left(|x-y|\right)&\left[(1+|z|)^{p(x)-1} + (1+|z|)^{p(y)-1}\right] \\ &\times\left[1+\log\left(1+|z|\right)\right]. \end{aligned}$$

Here $\mu:[0,1)\to[0,\infty)$ is a monotone nondecreasing, square-concave modulus of continuity, satisfying $\mu(0)=0$. Both ω and $\omega_\xi:[0,\infty)\to[0,1]$ satisfy $\omega(t)\le\min\{t^\alpha,1\}$ for some fixed $\alpha\in(0,1)$.

The inhomogeneity b to satisfies either

(B1) Controllable growth: $b(x,\xi,z)\le L(1+|z|)^{p(x)-1}$,

or

(B2) Natural growth for bounded solutions: $b(x,\xi,z)\le L_1|z|^{p(x)} + L_2$,

for $L_1, L_2 > 0$ where $L_1 = L_1(\|u\|_{L^\infty})$ satisfies $2L_1\|u\|_{L^\infty} < \nu$.

Statement of main result

We are now in a position to state our main theorem.

Theorem 6.1. *Let $u\in g+W_0^{1,p(\,\cdot\,)}\big(\Omega,\mathbb{R}^N\big)$ be a weak solution to* (6.1) *under assumptions* **(H1)–(H6)***, where the inhomogeneity b satisfies either* **(B1)***, or* **(B2)** *under the additional assumption that the solution is bounded with $2L_1\|u\|_{L^\infty}<\nu$. Then the following hold:*

(i) $\mathrm{Reg}_{Du}(\overline{\Omega})$ *is relatively open in* $\overline{\Omega}$*,*

(ii) $u\in C^{1,\sigma}\big(\mathrm{Reg}_{Du}(\overline{\Omega}),\mathbb{R}^N\big)$ *for* $\sigma=\alpha$*,*

(iii) $\mathrm{Sing}_{Du}(\Omega)\subset\big(\Sigma_{1,\Omega}\cup\Sigma_{2,\Omega}\cup\Sigma_{1,\partial\Omega}\cup\Sigma_{2,\partial\Omega}\big)$*, where the sets are defined in* (2.6)*–*(2.9)*.*

In particular, we have $\mathcal{L}^n\big(\mathrm{Sing}_u(\Omega)\big)=0$.

A Caccioppoli inequality

A crucial step in this proof is the establishment of a suitable Caccioppoli, or reverse Poincare inequality. This family of estimates allows us to locally control the gradient of the solution in terms of the solution itself, at some suitable scale. The sharpness of the terms on the right hand side will become crucial when deducing the optimal Hölder exponent, which is in some sense inherited from this estimate. A novel feature of the following inequality is that, in contrast to similar results in the literature, we require only finiteness of $(Du)_{x_0,\rho}$ or $(D_n u)^+_{x_0,\rho} \otimes e_n$ rather than that of $(|Du|^{p(x)})^+_{x_0,\rho}$. This allows for a characterisation of the singular sets that is in line with the fixed exponent cases (see [Bec07, DG00, Ham07]).

Lemma 6.2 (Interior Caccioppoli Inequality)**.** *Let $M > 0$ and assume that $u \in W^{1,p(\,\cdot\,)}(\Omega, \mathbb{R}^N)$ is a weak solution to* (6.1) *under structure conditions* **(H1)**–**(H6)** *with the inhomogeneity satisfying either* **(B1)***, or* **(B2)** *under the additional assumption that the solution is bounded with $2L_1\|u\|_{L^\infty} < \nu$. Set $\xi \in \mathbb{R}^N$, and $\Lambda \in \mathbb{R}^{nN}$ such that $|\xi|, |\Lambda| \leq M$. Then there exist constants $\rho_1 = \rho_1(n, N, L/\nu, \gamma_1, \gamma_2, L_1, L_2, \omega_p, \|u\|_\infty, M) \leq \rho_0$ and $c = c_c(n, N, L/\nu, p_2, \gamma_1, \gamma_2, L_1, L_2, E, \omega_p, M)$ such that for every $\rho < \rho_1$ and any ball $B_\rho(x_0) \subset\subset \Omega$ with $p_2 = \sup_{B_\rho(x_0)} p(\,\cdot\,)$, and $(Du)_{x_0,\rho} \leq M$ with $\mathcal{C}(x_0, (Du)_{x_0,\rho}, \rho) \leq \frac{1}{36}$, the following estimate holds:*

$$\fint_{B_{\frac{\rho}{2}}(x_0)} |V(Du - \Lambda)|^2\, dx \leq c_c \Bigg(\fint_{B_\rho(x_0)} \left| V\left(\frac{u - \xi - \Lambda(x - x_0)}{\rho} \right) \right|^2 dx \\ + \fint_{B_\rho(x_0)} \omega_\xi^2(|u - \xi|)\, dx + \rho^\alpha \Bigg).$$

Here of course, $V = V_{p_2}$. We also have the corresponding result on half balls, capturing the estimate for boundary points.

Lemma 6.3 (Boundary Caccioppoli Inequality)**.** *Let $M > 0$ and assume that $u \in W^{1,p(\,\cdot\,)}_\Gamma(B^+, \mathbb{R}^N)$ is a weak solution to* (6.2) *under structure conditions* **(H1)**–**(H6)** *with the inhomogeneity satisfying either* **(B1)***, or* **(B2)** *under the additional assumption that the solution is bounded with $2L_1\|u\|_{L^\infty} < \nu$. Choose $x_0 \in \Gamma$, and $\xi \in \mathbb{R}^N$ satisfying $|\xi| \leq M$. There exists a $\rho_1 = \rho_1(n, N, L/\nu, \gamma_1, \gamma_2, L_1, L_2, \omega_p, \|u\|_\infty, M) \leq \rho_0$ and a constant $c = c_c(n, N, L/\nu, p_2, \gamma_1, \gamma_2, L_1, L_2, E, \omega_p, M)$ such that whenever $\rho < \min\{\rho_1, 1 - |x_0|\}$, and $|\,(D_n u)^+_{x_0,\rho} \otimes e_n| \leq M$ with $\mathcal{C}(x_0, (D_n u)^+_{x_0,\rho} \otimes e_n, \rho) \leq \frac{1}{36}$, then the following estimate* *holds:*

$$\fint_{B^+_{\frac{\varrho}{2}}(x_0)} |V(Du - \xi \otimes e_n)|^2 \, dx \le c_c \Bigg(\fint_{B^+_\rho(x_0)} \left| V\left(\frac{u - \xi x_n}{\rho}\right)\right|^2 \, dx$$

$$+ \fint_{B^+_\rho(x_0)} \omega_\xi^2(|u - \xi x_n|) \, dx + \rho^\alpha \Bigg).$$

We present here the proof for the boundary version. The interior version is essentially analogous, and we will comment on specific differences.

Proof of Lemma 6.3: Taking a standard cutoff function $\eta \in C_0^\infty(B_\rho(x_0))$ satisfying $0 \le \eta \le 1$, $\eta = 1$ on $B_{\frac{\varrho}{2}}(x_0)$, $\eta = 0$ outside $B_{\frac{3\rho}{4}}(x_0)$ and $|D\eta| \le \frac{C}{\rho}$, we write $\phi := \eta^{\hat{p}} w$ where $\hat{p} = \max\{2, p_2\}$ and $w := u - \xi x_n$. Note that $\phi \in W^{1,p(\cdot)}_\Gamma(B^+_{\rho_0}(0), \mathbb{R}^N)$, with

$$D\phi = \hat{p}\eta^{\hat{p}-1}\big(u - \xi x_n\big) \otimes D\eta + \eta^{\hat{p}}(Du - \xi \otimes e_n). \tag{6.3}$$

In the interior case we of course take $\eta \in C_0^\infty(B_\rho(x_0))$ satisfying $0 \le \eta \le 1$, $\eta = 1$ on $B_{\frac{\varrho}{2}}(x_0)$, $\eta = 0$ outside $B_{\frac{3\rho}{4}}(x_0)$ and $|D\eta| \le \frac{C}{\rho}$. We again write $\phi := \eta^{\hat{p}} w$ for $\hat{p} = \max\{2, p_2\}$ and $w := u - \xi - \Lambda(x - x_0)$. Then $\phi \in W^{1,p(\cdot)}_0(B_\rho(x_0), \mathbb{R}^N)$, with

$$D\phi = \hat{p}\eta^{\hat{p}-1}(u - \xi \ - \Lambda(x - x_0)) \otimes D\eta + \eta^{\hat{p}}(Du - \Lambda). \tag{6.4}$$

Since u solves (6.2) and taking some point $\hat{x} \in \overline{B^+_\rho(x_0)}$ where $p(\hat{x}) = p_2$, there holds

$$\fint_{B^+_\rho(x_0)} a(x, u, Du) \cdot D\phi \, dx = \fint_{B^+_\rho(x_0)} b(x, u, Du) \cdot \phi \, dx,$$

and trivially

$$\fint_{B^+_\rho(x_0)} a(\hat{x}, 0, \Lambda) \cdot D\phi \, dx = 0.$$

For choice of $\Lambda = \xi \otimes e_n$, we proceed to calculate

$$\begin{aligned}
&\fint_{B_\rho^+(x_0)} \eta^{\hat{p}} \left[a(\hat{x}, u, Du) - a(\hat{x}, u, \Lambda)\right] \cdot (Du - \Lambda) \, dx \\
&\qquad = \fint_{B_\rho^+(x_0)} \left[a(\hat{x}, u, Du) - a(x, u, Du)\right] \cdot D\phi \, dx \\
&\qquad\quad + \fint_{B_\rho^+(x_0)} \left[a(\hat{x}, \xi x_n, \Lambda) - a(\hat{x}, u, \Lambda)\right] \cdot D\phi \, dx \\
&\qquad\quad + \fint_{B_\rho^+(x_0)} \left[a(\hat{x}, 0, \Lambda) - a(\hat{x}, \xi x_n, \Lambda)\right] \cdot D\phi \, dx \\
&\qquad\quad + \hat{p} \fint_{B_\rho^+(x_0)} \eta^{\hat{p}-1} \left[a(\hat{x}, u, \Lambda) - a(\hat{x}, u, Du)\right] \cdot w \otimes D\eta \, dx \\
&\qquad\quad + \fint_{B_\rho^+(x_0)} b(x, u, Du) \cdot \eta^{\hat{p}} w,
\end{aligned}$$

with the obvious labelling

$$\mathrm{I} = \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI}.$$

Note that in the interior case, we replace ξx_n with ξ, and term IV does not appear. We now consider each term independently. From **(H1)** and elementary integration we can use Lemma 3.4 (*iv*) to estimate for the case $2 \leq p_2 < \infty$ that

$$\begin{aligned}
\mathrm{I} &= \fint_{B_\rho^+(x_0)} \eta^{\hat{p}} \left[a(\hat{x}, u, Du) - a(\hat{x}, u, \Lambda)\right] \cdot (Du - \Lambda) \, dx \\
&= \fint_{B_\rho^+(x_0)} \eta^{\hat{p}} \int_0^1 \left[D_z a(\hat{x}, u, \Lambda + t(Du - \Lambda))(Du - \Lambda)\right] \cdot (Du - \Lambda) \, dt \, dx \\
&\geq c \fint_{B_\rho^+(x_0)} \eta^{\hat{p}} \int_0^1 \left(1 + |\Lambda + t(Du - \Lambda)|^2\right)^{\frac{p_2 - 2}{2}} |Du - \Lambda|^2 \, dt \, dx \\
&\geq c \fint_{B_\rho^+(x_0)} \eta^{\hat{p}} |Du - \Lambda|^2 \, dx \\
&\geq c \fint_{B_\rho^+(x_0)} \eta^{\hat{p}} \left|V\left(Du - \Lambda\right)\right|^2 \, dx.
\end{aligned}$$

Here the constant depends only on ν and p_2. On the other hand, when p_2 is subquadratic we again use **(H1)** and elementary integration, with Corollary 4.3 (*ii*), our definition

(3.1) of the function V, and the bound $|\,(D_n u)^+_{x_0,\rho}\otimes e_n|\le M$ to find

$$\begin{aligned}
\mathrm{I} &= \fint_{B_\rho^+(x_0)} \eta^{\hat p}\int_0^1 \left[D_z a(\hat x,u,\Lambda+t(Du-\Lambda))(Du-\Lambda)\right]\,\cdot\,(Du-\Lambda)\;dt\,dx\\
&\ge c\fint_{B_\rho^+(x_0)} \eta^{\hat p}\int_0^1 \left(1+|\Lambda+t(Du-\Lambda)|^2\right)^{\frac{p_2-2}{2}}|Du-\Lambda|^2\,dt\,dx\\
&\ge c\fint_{B_\rho^+(x_0)} \eta^{\hat p}\int_0^1 \left(1+2|\Lambda|^2+2t^2|Du-\Lambda|^2\right)^{\frac{p_2-2}{2}}|Du-\Lambda|^2\,dt\,dx\\
&\ge c(1+|\Lambda|)^{p_2-2}\fint_{B_\rho^+(x_0)} \eta^{\hat p}\left(1+|Du-\Lambda|^2\right)^{\frac{p_2-2}{2}}|Du-\Lambda|^2\,dx\\
&\ge c(1+|\,(D_n u)^+_{x_0,\rho}\otimes e_n|)^{p_2-2}\fint_{B_\rho^+(x_0)} \eta^{\hat p}\left|V\left(Du-\Lambda\right)\right|^2\;dx \qquad (6.5)\\
&\ge c\fint_{B_\rho^+(x_0)} \eta^{\hat p}\left|V\left(Du-\Lambda\right)\right|^2\;dx, \qquad (6.6)
\end{aligned}$$

where the constant depends on ν, M and p_2.

Considering the second term, we calculate via (6.4) that

$$\begin{aligned}
\mathrm{II} &= \fint_{B_\rho^+(x_0)} \left[a(\hat x,u,Du)-a(x,u,Du)\right]\,\cdot\,D\phi\,dx\\
&= \fint_{B_\rho^+(x_0)} \hat p\eta^{\hat p-1}\left[a(\hat x,u,Du)-a(x,u,Du)\right]\,\cdot\,w\otimes D\eta\,dx\\
&\qquad+\fint_{B_\rho^+(x_0)} \eta^{\hat p}\left[a(\hat x,u,Du)-a(x,u,Du)\right]\,\cdot\,(Du-\Lambda)\,dx\\
&= \mathrm{II}_a+\mathrm{II}_b,
\end{aligned}$$

again with the obvious labelling.

Now we can use **(H6)** and the monotonicity of ω, then Young's inequality to compute

$$\begin{aligned}
\mathrm{II}_b &\le \fint_{B_\rho^+(x_0)} \eta^{\hat p}\big|a(\hat x,u,Du)-a(x,u,Du)\big|\big|Du-\Lambda\big|\,dx\\
&\le 2L\fint_{B_\rho^+(x_0)} \eta^{\hat p}\omega\big(|x-\hat x|\big)\big(1+|Du|^2\big)^{\frac{p_2-1}{2}}\Big[1+\log(1+|Du|)\Big]|Du-\Lambda|\,dx\\
&\le c\fint_{B_\rho^+(x_0)} \eta^{\hat p}\omega(2\rho)\big(1+|Du|^2\big)^{\frac{p_2}{4}+\frac{p_2-2}{4}}\Big[1+\log(1+|Du|)\Big]|Du-\Lambda|\,dx\\
&\le c(\varepsilon)\omega^{\hat p}(2\rho)\fint_{B_\rho^+(x_0)} \big(1+|Du|^2\big)^{\frac{p_2}{2}}\Big[1+\log^2(1+|Du|)\Big]\,dx\\
&\qquad+\varepsilon\fint_{B_\rho^+(x_0)} \eta^{\hat p}\big(1+|Du|^2\big)^{\frac{p_2-2}{2}}|Du-\Lambda|^2\,dx.
\end{aligned}$$

At this point the first integral is controlled by the elementary inequality $\log(1+z)\le$

$C(\delta)z^{\delta}$ followed by Corollary 4.2 with $p_0 = p_2(1+\frac{\hat{\delta}}{4})$, and Corollary 4.3 (ii). The second term follows by Lemma 3.4 (iv), since

$$\begin{aligned}
c(\varepsilon)\omega^{\hat{p}}(2\rho) &\fint_{B_\rho^+(x_0)} \big(1+|Du|^2\big)^{\frac{p_2}{2}} \log^2(1+|Du|)\,dx + \varepsilon \fint_{B_\rho^+(x_0)} \eta^{\hat{p}}\big(1+|Du|^2\big)^{\frac{p_2-2}{2}}|Du-\Lambda|^2\,dx \\
&\le c(\varepsilon,\delta)\omega^2(2\rho)\fint_{B_\rho^+(x_0)} 1+|Du|^{p_2(1+\frac{\delta}{4})}\,dx + \varepsilon\fint_{B_\rho^+(x_0)} \eta^{\hat{p}}\big|V\big(Du-\Lambda\big)\big|^2\,dx \\
&\le c(\varepsilon,\delta)\omega^2(2\rho)\bigg(\fint_{B_\rho^+(x_0)} 1+|Du|^{p_2}\,dx\bigg)^{\frac{1}{1+\frac{\delta}{4}}} + \varepsilon\fint_{B_\rho^+(x_0)} \eta^{\hat{p}}\big|V\big(Du-\Lambda\big)\big|^2\,dx \\
&\le c(\varepsilon,\delta)\omega^2(2\rho)\big(1+|\,(D_nu)^+_{x_0,\rho}\otimes e_n|\big)^{\frac{1}{1+\frac{\delta}{4}}} + \varepsilon\fint_{B_\rho^+(x_0)} \eta^{\hat{p}}\big|V\big(Du-\Lambda\big)\big|^2\,dx \\
&\le \varepsilon\fint_{B_\rho^+(x_0)} \eta^{\hat{p}}\big|V\big(Du-\Lambda\big)\big|^2\,dx + c(\varepsilon,\delta,M)\omega^2(\rho). \qquad (6.7)
\end{aligned}$$

Here, we can choose ε, say $\varepsilon = \frac{c}{64}$ where c is the constant from (6.5), so that it becomes small enough to be absorbed by the lower bound on I. We note that use of Corollary 4.2 and Corollary 4.3 with $|\,(D_nu)^+_{x_0,\rho}\otimes e_n| \le M$ puts $c = c(n,N,p_2,L/\nu,\gamma_1,\gamma_2,L_1,L_2,E,\omega_p,M)$. The constant will gain no additional dependence, so they will not be mentioned again.

Estimating II_a, we first consider the case where $2 \le p_2 < \infty$. We use **(H6)** and the monotonicity of ω, then Young's inequality to split the integrand into two terms. Then the inequality $\log(1+z) \le C(\delta)z^{\delta}$, Corollary 4.2, and Corollary 4.3 let us control the first term in a similar manner to (6.7). Again, Lemma 3.4 (iv) gives us the latter estimate

$$\begin{aligned}
\mathrm{II}_a &\le c\fint_{B_\rho^+(x_0)} |a(\hat{x},u,Du)-a(x,u,Du)|\left|\frac{w}{\rho}\right|dx \\
&\le c\fint_{B_\rho^+(x_0)} \omega(|x-\hat{x}|)\big(1+|Du|^2\big)^{\frac{p_2-1}{2}}\Big[1+\log(1+|Du|)\Big]\left|\frac{w}{\rho}\right|dx \\
&\le c\omega^{\frac{p_2}{p_2-1}}(2\rho)\fint_{B_\rho^+(x_0)} \big(1+|Du|^2\big)^{\frac{p_2}{2}}\Big[1+\log^{\frac{p_2}{p_2-1}}(1+|Du|)\Big]\,dx + c\fint_{B_\rho^+(x_0)}\left|\frac{w}{\rho}\right|^{p_2}dx \\
&\le c\omega^{\frac{p_2}{p_2-1}}(2\rho) + c\fint_{B_\rho^+(x_0)}\left|V\Big(\frac{w}{\rho}\Big)\right|^2 dx.
\end{aligned}$$

We have used the assumption $|\,(D_nu)^+_{x_0,\rho}\otimes e_n| \le M$ in the last step. For the subquadratic case, we split the domain $B_\rho(x_0)$ into two sets,

$$B_- := \left\{x\in B_\rho^+(x_0) : \left|\frac{w}{\rho}\right| < 1\right\} \qquad \text{and} \qquad B_+ = B_\rho^+(x_0)\setminus B_-.$$

On B_-, we note that for $1 < p_2 < 2$ we always have $p_2 - 1 < \frac{p_2}{2}$. In a similar way, we use **(H6)**, the monotonicity of ω, and Young's inequality. The first term is again estimated using Corollary 4.2 and Corollary 4.3, and the second by Lemma 3.4 (iv)

$$
\begin{aligned}
\int_{B_-} & \hat{p}\eta^{\hat{p}-1}\big[a(\hat{x},u,Du)-a(x,u,Du)\big]\cdot w\otimes D\eta\,dx \\
&\le c\int_{B_-}\big|a(\hat{x},u,Du)-a(x,u,Du)\big|\left|\frac{w}{\rho}\right|dx \\
&\le c\int_{B_-}\omega(|x-\hat{x}|)\big(1+|Du|^2\big)^{\frac{p_2-1}{2}}\Big[1+\log(1+|Du|)\Big]\left|\frac{w}{\rho}\right|dx \\
&\le c\omega^2(2\rho)\int_{B_\rho^+(x_0)}\big(1+|Du|^2\big)^{p_2-1}\Big[1+\log^2(1+|Du|)\Big]\,dx+c\int_{B_\rho^+(x_0)}\left|\frac{w}{\rho}\right|^2dx \\
&\le c\omega^2(2\rho)+c\int_{B_\rho^+(x_0)}\left|V\left(\frac{w}{\rho}\right)\right|^2dx.
\end{aligned}
$$

Turning our attention to B_+ we use similar arguments, changing only the exponents in Young's inequality and using the fact that $\omega\le 1$ to find

$$
\begin{aligned}
\int_{B_+} & \hat{p}\eta^{\hat{p}-1}\big[a(\hat{x},u,Du)-a(x,u,Du)\big]\cdot w\otimes D\eta\,dx \\
&\le c\int_{B_+}\big|a(\hat{x},u,Du)-a(x,u,Du)\big|\left|\frac{w}{\rho}\right|dx \\
&\le c\int_{B_+}\omega(|x-\hat{x}|)\big(1+|Du|^2\big)^{\frac{p_2-1}{2}}\Big[1+\log(1+|Du|)\Big]\left|\frac{w}{\rho}\right|dx \\
&\le c\omega^{\frac{p_2}{p_2-1}}(2\rho)\int_{B_\rho^+(x_0)}\big(1+|Du|^2\big)^{\frac{p_2}{2}}\Big[1+\log^{\frac{p_2}{p_2-1}}(1+|Du|)\Big]\,dx+c\int_{B_\rho^+(x_0)}\left|\frac{w}{\rho}\right|^{p_2}dx \\
&\le c\omega^2(2\rho)+c\int_{B_\rho^+(x_0)}\left|V\left(\frac{w}{\rho}\right)\right|^2dx.
\end{aligned}
$$

Recombining these sets and taking averages over the domain, recalling $\omega\le 1$ we arrive at

$$
\begin{aligned}
\mathrm{II}_a &\le c\omega^{\frac{p_2}{p_2-1}}(2\rho)+c\omega^2(2\rho)+c\fint_{B_\rho^+(x_0)}\left|V\left(\frac{w}{\rho}\right)\right|^2dx \\
&\le c\omega(2\rho)+c\fint_{B_\rho^+(x_0)}\left|V\left(\frac{w}{\rho}\right)\right|^2dx.
\end{aligned}
$$

To estimate III we notice that owing to **(H5)** there holds

$$\begin{aligned}
\mathrm{III} &= \fint_{B_\rho^+(x_0)} \big[a(\hat{x},\xi x_n,\Lambda) - a(\hat{x},u,\Lambda)\big] \cdot D\phi\,dx \\
&= \fint_{B_\rho^+(x_0)} \hat{p}\eta^{\hat{p}-1}\big[a(\hat{x},\xi x_n,\Lambda) - a(\hat{x},u,\Lambda)\big] \cdot w\otimes D\eta\,dx \\
&\quad + \fint_{B_\rho^+(x_0)} \eta^{\hat{p}}\big[a(\hat{x},\xi x_n,\Lambda) - a(x,u,\Lambda)\big] \cdot (Du-\Lambda)\,dx \\
&\le c\fint_{B_\rho^+(x_0)} \big|a(\hat{x},\xi x_n,\Lambda) - a(\hat{x},u,\Lambda)\big|\left|\frac{w}{\rho}\right| dx \\
&\quad + \fint_{B_\rho^+(x_0)} \eta^{\hat{p}}\big|a(\hat{x},\xi x_n,\Lambda) - a(x,u,\Lambda)\big|\big|Du-\Lambda\big|\,dx \\
&\le c\fint_{B_\rho^+(x_0)} \omega_\xi\big(|u-\xi x_n|\big)\big(1+|\Lambda|^2\big)^{\frac{p(x)-1}{2}}\left|\frac{w}{\rho}\right| dx \\
&\quad + c\fint_{B_\rho^+(x_0)} \eta^{\hat{p}}\omega_\xi\big(|u-\xi x_n|\big)\big(1+|\Lambda|^2\big)^{\frac{p(x)-1}{2}}\big|Du-\Lambda\big|\,dx \\
&\le c\fint_{B_\rho^+(x_0)} \omega_\xi\big(|u-\xi x_n|\big)\left|\frac{w}{\rho}\right| dx + c\fint_{B_\rho^+(x_0)} \eta^{\hat{p}}\omega_\xi\big(|u-\xi x_n|\big)\big|Du-\Lambda\big|\,dx \\
&= \mathrm{III}_a + \mathrm{III}_b,
\end{aligned}$$

with the obvious notation. In the last step we have used Corollary 4.3 and the assumption that $|\,(D_n u)^+_{x_0,\rho}\otimes e_n| \le M$ to absorb those terms into the constant. Now when $2 \le p_2 < \infty$ we can use Young's inequality and Lemma 3.4 (iv) to deduce

$$\begin{aligned}
\mathrm{III}_a + \mathrm{III}_b &\le c(\varepsilon)\fint_{B_\rho^+(x_0)} \omega_\xi^2\big(|u-\xi x_n|\big) + \left|\frac{w}{\rho}\right|^2 dx + \varepsilon\fint_{B_\rho^+(x_0)} \eta^{2\hat{p}}|Du-\Lambda|^2\,dx \\
&\le c(\varepsilon)\fint_{B_\rho^+(x_0)} \omega_\xi^2\big(|u-\xi x_n|\big)\,dx + c\fint_{B_\rho^+(x_0)} \left|V\left(\frac{w}{\rho}\right)\right|^2 dx \\
&\qquad + \varepsilon\fint_{B_\rho^+(x_0)} \eta^{\hat{p}}\big|V\big(Du-\Lambda\big)\big|^2\,dx.
\end{aligned}$$

To treat the subquadratic case, we decompose our domain into the sets

$$C_- := \big\{x\in B_\rho^+(x_0) : |Du-\Lambda| < 1\big\} \qquad \text{and} \qquad C_+ = B_\rho^+(x_0)\setminus C_-.$$

We will consider III_b, with the calculations for III_a being completely analogous, replacing C_- with B_-, and C_+ with B_+. For $|Du-\Lambda| < 1$ we find owing again to Young's inequality and Lemma 3.4 (iv)

$$
\begin{aligned}
c\int_{C_-} &\eta^2\omega_\xi\big(|u-\xi x_n|\big)|Du-\Lambda|\,dx \\
&\le c(\varepsilon)\int_{C_-}\omega_\xi^2\big(|u-\xi x_n|\big)\,dx+\varepsilon\int_{C_-}\eta^4|Du-\Lambda|^2\,dx \\
&\le c(\varepsilon)\int_{B_\rho^+(x_0)}\omega_\xi^2\big(|u-\xi x_n|\big)\,dx+\varepsilon\int_{B_\rho^+(x_0)}\eta^{\hat p}\big|V(Du-\Lambda)\big|^2\,dx.
\end{aligned}
$$

On C_+ we can recall that $\omega_\xi\le 1$ and carefully note that $\frac{p_2}{p_2-1}>2$ to similarly obtain

$$
\begin{aligned}
c\int_{C_-} &\eta^2\omega_\xi\big(|u-\xi x_n|\big)|Du-\Lambda|\,dx \\
&\le c(\varepsilon)\int_{C_-}\omega_\xi^{\frac{p_2}{p_2-1}}\big(|u-\xi x_n|\big)\,dx+\varepsilon\int_{C_-}\eta^{2p_2}|Du-\Lambda|^{p_2}\,dx \\
&\le c(\varepsilon)\int_{B_\rho^+(x_0)}\omega_\xi^2\big(|u-\xi x_n|\big)\,dx+\varepsilon\int_{B_\rho^+(x_0)}\eta^{\hat p}\big|V(Du-\Lambda)\big|^2\,dx.
\end{aligned}
$$

Combining these sets, and taking averages over the domain of integration gives

$$
\mathrm{III}\le c(\varepsilon)\fint_{B_\rho^+(x_0)}\omega_\xi^2\big(|u-\xi x_n|\big)\,dx+c\fint_{B_\rho^+(x_0)}\left|V\left(\frac{w}{\rho}\right)\right|^2dx+\varepsilon\fint_{B_\rho^+(x_0)}\eta\big|V(Du-\Lambda)\big|^2\,dx.
$$

The calculations for IV are completely analogous to those for III, changing only the second argument of a, and noting that $|\xi x_n|\le M\rho$ to compute

$$
\mathrm{IV}\le c(\varepsilon)\omega_\xi^2\big(M\rho\big)+c\fint_{B_\rho^+(x_0)}\left|V\left(\frac{w}{\rho}\right)\right|^2dx+\varepsilon\fint_{B_\rho^+(x_0)}\eta\big|V(Du-\Lambda)\big|^2\,dx.
$$

Estimating the next term in the case where $p_2\ge 2$, we again use basic integration in order to exploit condition **(H3)**. Corollary 4.3 with the bound $|\,(D_nu)^+_{x_0,\rho}\otimes e_n|\le M$, Young's inequality and Lemma 3.4 (iv) then let us calculate

$$\begin{aligned}
\mathrm{V} &= \hat{p} \fint_{B_\rho^+(x_0)} \eta^{\hat{p}-1} \big[a(\hat{x}, u, \Lambda) - a(\hat{x}, u, Du)\big] \cdot w \otimes D\eta \, dx \\
&= \hat{p} \fint_{B_\rho^+(x_0)} \eta^{\hat{p}-1} \int_0^1 D_z a\big(\hat{x}, u, \Lambda + t(Du - \Lambda)\big)\big(Du - \Lambda\big) \cdot w \otimes D\eta \, dt \, dx \\
&\leq c \fint_{B_\rho^+(x_0)} \eta^{\hat{p}-1} \int_0^1 \big|D_z a\big(\hat{x}, u, \Lambda + t(Du - \Lambda)\big)\big| |Du - \Lambda| |w| |D\phi| \, dt \, dx \\
&\leq c \fint_{B_\rho^+(x_0)} \eta^{\hat{p}-1} \int_0^1 \big(1 + \big|\Lambda + t(Du - \Lambda)\big|^2\big)^{\frac{p_2-2}{2}} |Du - \Lambda| \left|\frac{w}{\rho}\right| dt \, dx \\
&\leq c \fint_{B_\rho^+(x_0)} \eta^{\hat{p}-1} \int_0^1 \big(1 + |\Lambda|^2 + |Du - \Lambda|^2\big)^{\frac{p_2-2}{2}} |Du - \Lambda| \left|\frac{w}{\rho}\right| dt \, dx \\
&\leq c \fint_{B_\rho^+(x_0)} \eta^{\hat{p}-1} \big(1 + |Du - \Lambda|^2\big)^{\frac{p_2-2}{2}} |Du - \Lambda| \left|\frac{w}{\rho}\right| dx \\
&\leq c \fint_{B_\rho^+(x_0)} \eta^{\hat{p}-1} |Du - \Lambda| \left|\frac{w}{\rho}\right| dx + c \fint_{B_\rho^+(x_0)} \eta^{\hat{p}-1} |Du - \Lambda|^{p_2-1} \left|\frac{w}{\rho}\right| dx \\
&\leq c\varepsilon \fint_{B_\rho^+(x_0)} \eta^{2(\hat{p}-1)} |Du - \Lambda|^2 + \eta^{\hat{p}} |Du - \Lambda|^{p_2} \, dx + c(\varepsilon) \fint_{B_\rho^+(x_0)} \left|\frac{w}{\rho}\right|^2 + \left|\frac{w}{\rho}\right|^{p_2} dx \\
&\leq \varepsilon \fint_{B_\rho^+(x_0)} \eta^{\hat{p}} \big|V(Du - \Lambda)\big|^2 \, dx + c(\varepsilon) \fint_{B_\rho^+(x_0)} \left|V\left(\frac{w}{\rho}\right)\right|^2 dx.
\end{aligned}$$

The setting where $1 < p_2 < 2$ is more delicate. We split $B_\rho(x_0)$ into four mutually disjoint domains, defining

$$B_{+-} := \left\{x \in B_\rho^+(x_0) : |Du - \Lambda| \geq 1 \text{ and } \left|\frac{w}{\rho}\right| < 1\right\},$$

with B_{++}, B_{--} and B_{-+} having natural respective definitions. On B_{++} we compute via

(H2), Young's inequality and Lemma 3.4 (*iv*)

$$
\begin{aligned}
\hat{p}\int_{B_{++}} &\eta^{\hat{p}-1}\big[a(\hat{x},u,\Lambda)-a(\hat{x},u,Du)\big]\cdot w\otimes D\eta\,dx\\
&= 2\int_{B_{++}}\eta\big[a(\hat{x},u,\Lambda)-a(\hat{x},u,Du)\big]\cdot w\otimes D\eta\,dx\\
&\le 2\int_{B_{++}}\eta\big(|a(\hat{x},u,\Lambda)|+|a(\hat{x},u,Du)|\big)\left|\frac{w}{\rho}\right|dx\\
&\le c\int_{B_{++}}\eta\Big[\big(1+|Du|^2\big)^{\frac{p_2-1}{2}}+\big(1+|\Lambda|^2\big)^{\frac{p_2-1}{2}}\Big]\left|\frac{w}{\rho}\right|dx\\
&\le c\int_{B_{++}}\eta\big(1+|Du|^{p_2-1}+|\Lambda|^{p_2-1}\big)\left|\frac{w}{\rho}\right|dx\\
&\le c\int_{B_{++}}\eta\big(1+|Du-\Lambda|^{p_2-1}+|\Lambda|^{p_2-1}\big)\left|\frac{w}{\rho}\right|dx\\
&\le c\int_{B_{++}}\eta|Du-\Lambda|^{p_2-1}\left|\frac{w}{\rho}\right|dx\\
&\le \varepsilon\int_{B_{++}}\eta^{\frac{p_2}{p_2-1}}\,|Du-\Lambda|^{p_2}\,dx+c(\varepsilon)\int_{B_{++}}\left|\frac{w}{\rho}\right|^{p_2}dx\\
&\le \varepsilon\int_{B_\rho^+(x_0)}\eta^{\hat{p}}\big|V(Du-\Lambda)\big|^2dx+c(\varepsilon)\int_{B_\rho^+(x_0)}\left|V\left(\frac{w}{\rho}\right)\right|^2dx.
\end{aligned}
$$

On B_{+-} we similarly calculate from the fifth inequality above, changing only the exponents in Young's inequality

$$
\begin{aligned}
\hat{p}\int_{B_{+-}} &\eta^{\hat{p}-1}\big[a(\hat{x},u,\Lambda)-a(\hat{x},u,Du)\big]\cdot w\otimes D\eta\,dx\\
&\le c\int_{B_{+-}}\eta|Du-\Lambda|^{p_2-1}\left|\frac{w}{\rho}\right|dx\\
&\le \varepsilon\int_{B_{+-}}\eta^2\big|Du-\Lambda\big|^{2(p_2-1)}dx+c\int_{B_{+-}}\left|\frac{w}{\rho}\right|^2dx\\
&\le \varepsilon\int_{B_{+-}}\eta^2|Du-\Lambda|^{p_2}\,dx+c(\varepsilon)\int_{B_{+-}}\left|\frac{w}{\rho}\right|^2dx\\
&\le \varepsilon\int_{B_\rho^+(x_0)}\eta^{\hat{p}}|V(Du-\Lambda)|^2\,dx+c(\varepsilon)\int_{B_\rho^+(x_0)}\left|V\left(\frac{w}{\rho}\right)\right|^2dx.
\end{aligned}
$$

On B_{-+} we use elementary integration with **(H3)**, the fact that $p_2<2$ and Lemma 3.4

(iv) to see

$$
\begin{aligned}
2\int_{B_{-+}} & \eta\big[a(x_0,u,\Lambda)-a(x_0,u,Du)\big]\cdot w\otimes D\eta\,dx \\
&= 2\int_{B_{-+}}\eta\int_0^1 D_z a\big(x_0,u,Du+t(\Lambda-Du)\big)\big(\Lambda-Du\big)\cdot w\otimes D\eta\,dt\,dx \\
&\le c\int_{B_{-+}}\eta\int_0^1 \big|D_z a\big(x_0,u,Du+t(\Lambda-Du)\big)\big|\big|Du-\Lambda\big|\left|\frac{w}{\rho}\right|dt\,dx \\
&\le c\int_{B_{-+}}\eta\int_0^1 \big(1+|Du+t(\Lambda-Du)|^2\big)^{\frac{p_2-2}{2}}\big|Du-\Lambda\big|\left|\frac{w}{\rho}\right|dt\,dx \\
&\le c\int_{B_{-+}}\eta\big|Du-\Lambda\big|\left|\frac{w}{\rho}\right|dx \\
&\le c\int_{B_{-+}}\left|\frac{w}{\rho}\right|dx \\
&\le c\int_{B_{-+}}\left|\frac{w}{\rho}\right|^{p_2}dx \\
&\le c\int_{B_\rho^+(x_0)}\left|V\left(\frac{w}{\rho}\right)\right|^2dx.
\end{aligned}
$$

Finally, on B_{--} we calculate from the third inequality above via **(H3)**, Young's inequality and Lemma 3.4 (iv)

$$
\begin{aligned}
2\int_{B_{--}} & \eta\big(a(x_0,u,\Lambda)-a(x_0,u,Du)\big)\cdot w\otimes D\eta\,dx \\
&\le c\int_{B_{--}}\eta|Du-\Lambda|\left|\frac{w}{\rho}\right|dx \\
&\le \varepsilon\int_{B_{--}}\eta^2|Du-\Lambda|^2\,dx+c(\varepsilon)\int_{B_{--}}\left|\frac{w}{\rho}\right|^2dx \\
&\le \varepsilon\int_{B_\rho^+(x_0)}\eta^{\hat p}\big|V(Du-\Lambda)\big|^2\,dx+c(\varepsilon)\int_{B_\rho^+(x_0)}\left|V\left(\frac{w}{\rho}\right)\right|^2dx.
\end{aligned}
$$

Combining our domain and taking the average value gives

$$
\mathrm{V}\le\varepsilon⨍_{B_\rho^+(x_0)}\eta^{\hat p}\big|V(Du-\Lambda)\big|^2\,dx+c(\varepsilon)⨍_{B_\rho^+(x_0)}\left|V\left(\frac{w}{\rho}\right)\right|^2dx.
$$

When treating the inhomogeneous term, we first consider the case where b satisfies the controllable growth condition **(B1)**. In the superquadratic case, we estimate using this condition, Young's inequality with Lemma 3.4 (iv) and Corollaries 4.2 and 4.3 (i)

$$
\begin{aligned}
\mathrm{VI} &\leq L \fint_{B_\rho^+(x_0)} \rho\big(1+|Du|^2\big)^{\frac{p(x)-1}{2}} \eta^{\hat{p}} \left|\frac{w}{\rho}\right| dx \\
&\leq C \fint_{B_\rho^+(x_0)} \rho\big(1+|Du|^2\big)^{\frac{p_2-1}{2}} \left|\frac{w}{\rho}\right| dx \\
&\leq C \fint_{B_\rho^+(x_0)} \rho^{\frac{p_2}{p_2-1}}\big(1+|Du|^2\big)^{\frac{p_2}{2}} + \left|\frac{w}{\rho}\right|^{p_2} dx \\
&\leq C\left(\rho^{\frac{p_2}{p_2-1}} + \fint_{B_\rho^+(x_0)} \left|V\left(\frac{w}{\rho}\right)\right|^2 dx\right).
\end{aligned}
$$

In the subquadratic case, we consider two cases. Recalling the set B_+ from the estimates on II_a, we find the calculations are analogous to the superquadratic case. On its relative complement B_-, we change only the exponents in Young's inequality, keeping in mind $p_2 - 1 < \frac{p_2}{2}$ to find

$$
\begin{aligned}
\alpha_n \rho^n \mathrm{VI} &\leq L \int_{B_-} \rho\big(1+|Du|^2\big)^{\frac{p(x)-1}{2}} \eta^{\hat{p}} \left|\frac{w}{\rho}\right| dx \\
&\leq C \int_{B_-} \rho\big(1+|Du|^2\big)^{\frac{p_2-1}{2}} \left|\frac{w}{\rho}\right| dx \\
&\leq C \int_{B_-} \rho^2\big(1+|Du|^2\big)^{p_2-1} + \left|\frac{w}{\rho}\right|^2 dx \\
&\leq C \int_{B_-} \rho^2\big(1+|Du|^2\big)^{\frac{p_2}{2}} + \left|\frac{w}{\rho}\right|^2 dx \\
&\leq C\left(\int_{B_-} \rho^2 + \left|V\left(\frac{w}{\rho}\right)\right|^2 dx\right),
\end{aligned}
$$

whereupon averaging gives the required estimate.

Finally, we consider VI under the natural growth assumptions. We begin by noting that, for $\kappa \in (0,1)$ to be fixed later there holds

$$
\begin{aligned}
\mathrm{VI} &\leq \fint_{B_\rho^+(x_0)} \Big(L_1|Du|^{p(x)} + L_2\Big)\eta^{\hat{p}}|w|\, dx \\
&\leq L_1 \fint_{B_\rho^+(x_0)} \eta^{\hat{p}}\big(1+\kappa\big)^{p_2}|Du-\Lambda|^{p(x)}|w|\, dx \\
&\qquad + \Big(L_1\big(1+\kappa^{-1}\big)^{p_2}\big(1+|\Lambda|\big)^{p_2} + L_2\Big) \fint_{B_\rho^+(x_0)} \rho\left|\frac{w}{\rho}\right| dx \\
&= \mathrm{VI}_a + \mathrm{VI}_b,
\end{aligned}
$$

with the obvious labelling. We first consider the second term in the superquadratic case, where we find via Young's inequality and Lemma 3.4 (iv) that

$$\begin{aligned}\mathrm{VI}_b &\leq c_\kappa \fint_{B_\rho^+(x_0)} \rho^2 + \left|\frac{w}{\rho}\right|^2 dx \\ &\leq c_\kappa \left(\rho^2 + \fint_{B_\rho^+(x_0)} \left|V\left(\frac{w}{\rho}\right)\right|^2 dx\right).\end{aligned}$$

The subquadratic situation again requires us to consider separate domains: on B_- the calculations are analogous to those above, while on B_+ we change only the exponents in Young's inequality to deduce via Lemma 3.4 (iv)

$$\begin{aligned}\mathrm{VI}_b &\leq c_\kappa \fint_{B_\rho^+(x_0)} \rho^{\frac{p_2}{p_2-1}} + \left|\frac{w}{\rho}\right|^{p_2} dx \\ &\leq c_\kappa \left(\rho^2 + \fint_{B_\rho^+(x_0)} \left|V\left(\frac{w}{\rho}\right)\right|^2 dx\right).\end{aligned}$$

The remaining term requires further treatment, we again begin by considering subsets of integration. On C_+ from the calculations for III, we find independent of p_2 that

$$\begin{aligned}\alpha_n \rho^n \mathrm{VI}_a &= L_1 \int_{C_+} \eta^{\hat{p}} \big(1+\kappa\big)^{p_2} |Du - \Lambda|^{p(x)} |w| \, dx \\ &\leq L_1 \int_{C_+} \eta^{\hat{p}} \big(1+\kappa\big)^{p_2} |Du - \Lambda|^{p_2} \|w\|_{L^\infty} \, dx \\ &\leq L_1 \big(\|u\|_{L^\infty} + |\Lambda|\rho + |\xi|\big) \big(1+\kappa\big)^{p_2} \int_{B_\rho^+(x_0)} \eta^{\hat{p}} |Du - \Lambda|^{p_2} \, dx,\end{aligned}$$

which after reaveraging can be absorbed on the left for κ, ρ sufficiently small, provided $|\xi| \leq \|u\|_{L^\infty}$.

On the set C_- , we have for $p_2 \geq 2$ from Young's inequality and Lemma 3.4 (iv)

$$\begin{aligned}\alpha_n \rho^n \mathrm{VI}_a &= L_1 \int_{C_-} \eta^{\hat{p}} \big(1+\kappa\big)^{p_2} |Du - \Lambda|^{p(x)} \, |w| \, dx \\ &\leq c \int_{C_-} \rho \left|\frac{w}{\rho}\right| dx \\ &\leq c_\kappa \int_{C_-} \rho^2 + \left|\frac{w}{\rho}\right|^2 dx \\ &\leq c_\kappa \int_{B_\rho^+(x_0)} \rho^2 + \left|V\left(\frac{w}{\rho}\right)\right|^2 dx.\end{aligned}$$

When treating the subquadratic case, we are required to further decompose the domain of integration. Considering B_{-+} from our estimates of IV we again use Young's inequality and Lemma 3.4 (iv)

$$\begin{aligned}
\alpha_n\rho^n \mathrm{VI}_a &= L_1\int_{B_{-+}} \eta^{\hat{p}}\big(1+\kappa\big)^{p_2}|Du-\Lambda|^{p(x)}\,|w|\,dx\\
&= c\int_{B_{-+}} \rho\left|\frac{w}{\rho}\right|dx\\
&\le c\int_{B_{-+}} \rho^{\frac{p_2}{p_2-1}}+\left|\frac{w}{\rho}\right|^{p_2}dx\\
&\le c\int_{B_\rho^+(x_0)} \rho^2+\left|V\left(\frac{w}{\rho}\right)\right|^2dx.
\end{aligned}$$

Finally, the calculations on B_{--} are analogous to the superquadratic case on C_-. Collecting our terms and choosing ε small enough to be absorbed on the left, we have obtained

$$\begin{aligned}
\fint_{B^+_{\frac{\rho}{2}}(x_0)}\big|V(Du-\Lambda)\big|^2dx &\le \fint_{B_\rho^+(x_0)}\eta^{\hat{p}}\big|V(Du-\Lambda)\big|^2dx\\
&\le \mathrm{I}=\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}\\
&\le c\Bigg(\fint_{B_\rho^+(x_0)}\left|V\left(\frac{w}{\rho}\right)\right|^2dx+\fint_{B_\rho^+(x_0)}\omega_\xi^2\big(|u-\xi x_n|\big)\,dx\\
&\qquad +\omega^{\min\left\{2,\frac{p_2}{p_2-1}\right\}}(2\rho)+\omega_\xi^2(M\rho)+\rho^{\min\left\{2,\frac{p_2}{p_2-1}\right\}}\Bigg)\\
&\le c_c\Bigg(\fint_{B_\rho^+(x_0)}\left|V\left(\frac{w}{\rho}\right)\right|^2dx+\fint_{B_\rho^+(x_0)}\omega_\xi^2\big(|u-\xi x_n|\big)\,dx+\rho^\alpha\Bigg).
\end{aligned}$$

□

Corollary 6.4. *We can use Jensen's inequality then Poincaré's inequality, noting $u-\xi x_n$ vanishes on Γ, together with Corollary 4.3 (i) to conclude*

$$\begin{aligned}
\fint_{B_\rho^+(x_0)}\omega_\xi^2(|u-\xi x_n|)\,dx &\le \fint_{B_\rho^+(x_0)}\omega_\xi\big(|u-\xi x_n|\big)\,dx\\
&\le \omega_\xi\left(\fint_{B_\rho^+(x_0)}|u-\xi x_n|\,dx\right)\\
&\le \omega_\xi\left(\fint_{B_\rho^+(x_0)}\rho|Du|\,dx\right)\\
&\le \omega_\xi\left(\rho\fint_{B_\rho^+(x_0)}1+|Du|^{p_2}\,dx\right)\\
&\le c\omega_\xi(C\rho)\\
&\le c\rho^\alpha.
\end{aligned}$$

Combining with Lemma 6.3, this yields

$$\fint_{B^+_{\frac{\rho}{2}}(x_0)} \big|V(Du - \xi \otimes e_n)\big|^2 \, dx \le c_c \Bigg(\fint_{B^+_\rho(x_0)} \bigg|V\Big(\frac{u - \xi x_n}{\rho}\Big)\bigg|^2 \, dx + \rho^\alpha \Bigg).$$

A corresponding estimate holds for interior points. Fixing choice of $\xi = (u)_{x_0,\rho}$, *we can again use Jensen and Poincaré's inequalities to obtain*

$$\fint_{B_{\frac{\rho}{2}}(x_0)} \big|V(Du - \Lambda)\big|^2 \, dx \le c_c \Bigg(\fint_{B_\rho(x_0)} \bigg|V\Big(\frac{u - (u)_{x_0,\rho} - \Lambda(x - x_0)}{\rho}\Big)\bigg|^2 \, dx + \rho^\alpha \Bigg).$$

Remark 6.5. *If we further know Du to be bounded, then u itself is Lipschitz continuous, and this estimate can be refined to*

$$\begin{aligned}\fint_{B^+_\rho(x_0)} \omega^2_\xi\big(|u - \xi x_n|\big) \, dx &\le \fint_{B^+_\rho(x_0)} \omega^2_\xi(C\rho) \, dx \\ &\le c\omega^2_\xi(C\rho) \\ &\le c\rho^{2\alpha}.\end{aligned}$$

Furthermore, using the boundedness of Du, together with Young's inequality and of course Lemma 3.4 (iv), we estimate II_a *in the superquadratic case as*

$$\begin{aligned}\mathrm{II}_a &\le c \fint_{B^+_\rho(x_0)} \big|a(\hat{x}, u, Du) - a(x, u, Du)\big| \bigg|\frac{w}{\rho}\bigg| \, dx \\ &\le c \fint_{B^+_\rho(x_0)} \omega\big(|x - \hat{x}|\big)\big(1 + |Du|^2\big)^{\frac{p_2 - 1}{2}} \Big(1 + \log(1 + |Du|)\Big) \bigg|\frac{w}{\rho}\bigg| \, dx \\ &\le c \fint_{B^+_\rho(x_0)} \omega\big(|x - \hat{x}|\big) \bigg|\frac{w}{\rho}\bigg| \, dx \\ &\le c \fint_{B^+_\rho(x_0)} \omega^2(2\rho) \, dx + c \fint_{B_\rho(x_0)} \bigg|\frac{w}{\rho}\bigg|^2 \, dx \\ &\le c\rho^{2\alpha} + c \fint_{B^+_\rho(x_0)} \bigg|V\Big(\frac{w}{\rho}\Big)\bigg|^2 \, dx.\end{aligned}$$

Similarly, we can improve the estimates on V. Under the controllable growth assumption *in the superquadratic case, we now find*

$$
\begin{aligned}
\mathrm{V} &\leq L \fint_{B_\rho^+(x_0)} \rho\big(1+|Du|^2\big)^{\frac{p(x)-1}{2}} \eta^{\hat{p}} \left|\frac{w}{\rho}\right| dx \\
&\leq C \fint_{B_\rho^+(x_0)} \rho \left|\frac{w}{\rho}\right| dx \\
&\leq C \fint_{B_\rho^+(x_0)} \rho^2 + \left|\frac{w}{\rho}\right|^2 dx \\
&\leq C\left(\rho^2 + \fint_{B_\rho^+(x_0)} \left|V\left(\frac{w}{\rho}\right)\right|^2 dx\right).
\end{aligned}
$$

Recollecting our terms, Lemma 6.3 now becomes

$$
\fint_{B_{\frac{\rho}{2}}^+(x_0)} \big|V(Du-\xi\otimes e_n)\big|^2 dx \leq c_c\left(\fint_{B_\rho^+(x_0)} \left|V\left(\frac{u-\xi x_n}{\rho}\right)\right|^2 dx + \rho^{2\alpha}\right),
$$

with interior version

$$
\fint_{B_{\frac{\rho}{2}}(x_0)} \big|V(Du-\Lambda)\big|^2 dx \leq c_c\left(\fint_{B_\rho(x_0)} \left|V\left(\frac{u-(u)_{x_0,\rho}-\Lambda(x-x_0)}{\rho}\right)\right|^2 dx + \rho^{2\alpha}\right).
$$

$\mathcal{A}$-harmonic approximation

The second step in the proof is to show that the solution to our PDE lies close to a solutions of a family of related linear PDE.

Lemma 6.6 (Interior Approximate $\mathcal{A}$-harmonicity)**.** *Fix $M>0$ and assume that u is a weak solution to* (6.1) *under structure conditions* **(H1)**–**(H6)***, with the inhomogeneity satisfying either* **(B1)** *or* **(B2)***, and fix $\beta = \frac{\alpha}{p_2}$. There exists a constant $c_1 = c_1(n,N,p_2,L/\nu,\gamma_1,\gamma_2,L_1,L_2,E,\omega_p)$ and a radius $\rho_0 << 1$ from Lemma 6.2 such that whenever $\rho<\rho_0$, $|\Lambda|, |\,(Du)_{x_0,\rho}\,| \leq M$ with $\mathcal{C}(x_0,\Lambda,\rho) \leq \frac{1}{16}$ there holds*

$$
\begin{aligned}
&\left|\fint_{B_\rho(x_0)} D_z a(\hat{x},0,\Lambda)\,(Du-\Lambda)\;\cdot D\varphi\, dx\right| \\
&\qquad\qquad \leq c_1\Big(\mu_M\big(\mathcal{C}^{\frac{1}{2}}(x_0,\Lambda,\rho)\big)\mathcal{C}^{\frac{1}{2}}(x_0,\Lambda,\rho) + \mathcal{C}(x_0,\Lambda,\rho) + \rho^\beta\Big)\|D\varphi\|_{C(B_\rho^+(x_0),\mathbb{R}^N)},
\end{aligned}
$$

for all $\varphi \in C_0^\infty(B_\rho(x_0),\mathbb{R}^N)$.

Lemma 6.7 (Boundary Approximate $\mathcal{A}$-harmonicity)**.** *Fix $M>0$ and assume that u is a weak solution to* (6.2)*, under structure conditions* **(H1)**–**(H6)** *with the inhomogeneity satisfying either* **(B1)** *or* **(B2)***, and fix $\beta = \frac{\alpha}{p_2}$. There exists a constant $c_1 = c_1(n,N,p_2,L/\nu,\gamma_1,\gamma_2,L_1,L_2,E,\omega_p)$ a radius $\rho_0 << 1$ from Lemma 6.2, such that* *whenever* $\rho < \rho_0 - |x_0|$ *for* $x_0 \in \Gamma_\rho(0)$, $|\Lambda|, |\,(D_n u)^+_{x_0,\rho} \otimes e_n| \le M$ *with* $\mathcal{C}(x_0,\Lambda,\rho) \le \frac{1}{16}$ *there holds*

$$\left| \fint_{B^+_\rho(x_0)} D_z a(\hat{x},0,\Lambda)\,(Du-\Lambda)\ \cdot D\varphi\,dx \right| \\ \le c_1 \Big(\mu_M\big(\mathcal{C}^{\frac12}(x_0,\Lambda,\rho)\big)\mathcal{C}^{\frac12}(x_0,\Lambda,\rho) + \mathcal{C}(x_0,\Lambda,\rho) + \rho^\beta \Big) \|D\varphi\|_{C(B^+_\rho(x_0),\mathbb{R}^N)},$$

for all $\varphi \in C^\infty_0(B_\rho(x_0),\mathbb{R}^N)$.

We will again present the proof for the boundary case, noting any arguments which differ for the interior estimates.

Proof of Lemma 6.7: Taking some $\varphi \in C^1_0(B^+_\rho(x_0),\mathbb{R}^N)$ with $\|D\varphi\|_{L^\infty(B^+_\rho(x_0),\mathbb{R}^N)} = 1$, we begin by noting that

$$\begin{aligned} &\fint_{B^+_\rho(x_0)} D_z a(\hat{x},0,\Lambda)(Du-\Lambda)\cdot D\varphi\,dx \\ &\qquad = \fint_{B^+_\rho(x_0)} \int_0^1 \big[D_z a(\hat{x},0,\Lambda) - D_z a\big(\hat{x},0,\Lambda + t(Du-\Lambda)\big)\big](Du-\Lambda)\cdot D\varphi\,dt\,dx \\ &\qquad\quad + \fint_{B^+_\rho(x_0)} a(\hat{x},0,Du)\cdot D\varphi\,dx \\ &\qquad = \fint_{B^+_\rho(x_0)} \int_0^1 \big[D_z a(\hat{x},0,\Lambda) - D_z a\big(\hat{x},0,\Lambda + t(Du-\Lambda)\big)\big](Du-\Lambda)\cdot D\varphi\,dt\,dx \\ &\qquad\quad + \fint_{B^+_\rho(x_0)} \big[a(\hat{x},0,Du) - a(x,0,Du)\big]\cdot D\varphi\,dx \\ &\qquad\quad + \fint_{B^+_\rho(x_0)} \big[a(x,0,Du) - a(x,(D_n u)^+_{x_0,\rho}\,x_n,Du)\big]\cdot D\varphi\,dx \\ &\qquad\quad + \fint_{B^+_\rho(x_0)} \big[a(x,(D_n u)^+_{x_0,\rho}\,x_n,Du) - a(x,u,Du)\big]\cdot D\varphi\,dx \\ &\qquad\quad + \fint_{B^+_\rho(x_0)} b(x,u,Du)\cdot\varphi\,dx \\ &\qquad = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}, \end{aligned}$$

with the obvious labelling. In the interior case we would simply replace the term $(D_n u)^+_{x_0,\rho}\,x_n$ by $(u)_{x_0,\rho} - (Du)_{x_0,\rho}\,(x-x_0)$.

Recall when estimating term III in Lemma 6.3 we defined the set $C_- = \{x \in B^+_\rho(x_0) : |Du-\Lambda| < 1\}$, and its relative complement in $B^+_\rho(x_0)$, C_+. To estimate I we first consider the set C_-. This allows us to compute via **(H4)**, Lemma 3.4 (*iv*), Hölder's inequality, and Jensen's inequality with the concavity of μ^2, then Hölder's inequality again that

$$\begin{aligned}
\mathrm{I}_- &= \frac{1}{\alpha_n\rho^n}\bigg|\int_{C_-}\int_0^1 \big[D_z a(\hat{x},0,\Lambda)-D_z a\big(\hat{x},0,\Lambda+t(Du-\Lambda)\big)\big](Du-\Lambda)\cdot D\varphi\,dt\,dx\bigg| \\
&\le \frac{1}{\alpha_n\rho^n}\int_{C_-}\int_0^1 \big|D_z a(\hat{x},0,\Lambda)-D_z a\big(\hat{x},0,\Lambda+t(Du-\Lambda)\big)\big|\big|Du-\Lambda\big|\,dt\,dx \\
&\le \frac{1}{\alpha_n\rho^n}\int_{C_-}\mu_M\big(|Du-\Lambda|\big)\big|V(Du-\Lambda)\big|\,dx \\
&\le \left(\frac{1}{\alpha_n\rho^n}\int_{C_-}\mu_M^2\big(|Du-\Lambda|\big)\,dx\right)^{\frac12}\left(\frac{1}{\alpha_n\rho^n}\int_{C_-}\big|V(Du-\Lambda)\big|^2\,dx\right)^{\frac12} \\
&\le \mu_M\!\left(\frac{1}{\alpha_n\rho^n}\int_{C_-}\big|Du-\Lambda\big|\,dx\right)\left(⨍_{B_\rho^+(x_0)}\big|V(Du-\Lambda)\big|^2\,dx\right)^{\frac12} \\
&\le \mu_M\!\left(\frac{1}{\alpha_n\rho^n}\int_{C_-}\big|V(Du-\Lambda)\big|\,dx\right)\mathcal{C}^{\frac12}(x_0,\Lambda,\rho) \\
&\le \mu_M\!\left(⨍_{B_\rho^+(x_0)}\big|V(Du-\Lambda)\big|\,dx\right)\mathcal{C}^{\frac12}(x_0,\Lambda,\rho) \\
&\le \mu_M\left(\mathcal{C}^{\frac12}(x_0,\Lambda,\rho)\right)\mathcal{C}^{\frac12}(x_0,\Lambda,\rho).
\end{aligned}$$

Similarly on the set C_+ we can make use of **(H3)** and **(H2)**, keeping in mind Corollary 4.3 and the bound $|\,(D_n u)^+_{x_0,\rho}\otimes e_n|\le M$,

$$\begin{aligned}
\mathrm{I}_+ &= \frac{1}{\alpha_n\rho^n}\bigg|\int_{C^+}\int_0^1 \big[D_z a(\hat{x},0,\Lambda)-D_z a\big(\hat{x},0,\Lambda+t(Du-\Lambda)\big)\big](Du-\Lambda)\,dt\cdot D\varphi\,dx\bigg| \\
&\le \frac{1}{\alpha_n\rho^n}\bigg|\int_{C^+}\big[D_z a(\hat{x},0,\Lambda)(Du-\Lambda)-a(\hat{x},0,Du)+a(\hat{x},0,\Lambda)\big]\cdot D\varphi\,dx\bigg| \\
&\le \frac{1}{\alpha_n\rho^n}\int_{C^+}\big|D_z a(\hat{x},0,\Lambda)\big|\big|Du-\Lambda\big|+\big|a(\hat{x},0,Du)\big|\,dx \\
&\le \frac{c}{\alpha_n\rho^n}\int_{C^+}\big(1+|\Lambda|^2\big)^{\frac{p(x)-2}{2}}|Du-\Lambda|+\big(1+|Du|^2\big)^{\frac{p_2-1}{2}}\,dx \\
&\le \frac{c(M)}{\alpha_n\rho^n}\int_{C^+}|Du-\Lambda|+\big(1+|\Lambda|^2+|Du-\Lambda|^2\big)^{\frac{p_2-1}{2}}\,dx \\
&\le \frac{c(M)}{\alpha_n\rho^n}\int_{C^+}|Du-\Lambda|+\big((1+|\,(D_n u)^+_{x_0,\rho}\otimes e_n|)^2+|Du-\Lambda|^2\big)^{\frac{p_2-1}{2}}\,dx
\end{aligned}$$

It i s now simple to estimate via Lemma 3.4 (iv) and our domain of definition

$$
\begin{aligned}
&\frac{c(M)}{\alpha_n \rho^n} \int_{C^+} |Du-\Lambda| + \big(1+|\Lambda|^2+|Du-\Lambda|^2\big)^{\frac{p_2-1}{2}}\,dx \\
&\qquad\qquad \le c(M)\frac{1}{\alpha_n\rho^n}\int_{C^+} |Du-\Lambda| + |Du-\Lambda|^{p_2-1}\,dx \\
&\qquad\qquad \le c(M)\frac{1}{\alpha_n\rho^n}\int_{C^+} |Du-\Lambda|^{p_2}\,dx \\
&\qquad\qquad \le c(M)\frac{1}{\alpha_n\rho^n}\fint_{B_\rho^+(x_0)} \big|V(Du-\Lambda)\big|^2\,dx \\
&\qquad\qquad = c(M)\,\mathcal{C}(x_0,\Lambda,\rho).
\end{aligned}
$$

Note that the first step works in both the superquadratic and subquadratic cases, since we can attain pointwise bounds on the first term by $(1+M^2)^{\frac{p_2-2}{2}}$ or 1, respectively. Combining these estimates gives

$$
\mathrm{I} \le \mathrm{I}_- + \mathrm{I}_+ \le \mu_M\left(\mathcal{C}^{\frac12}(x_0,\Lambda,\rho)\right)\mathcal{C}^{\frac12}(x_0,\Lambda,\rho) + c\,\mathcal{C}(x_0,\Lambda,\rho).
$$

Turning our attention to II, we employ **(H6)**, the sublinearity of log, and then Corollary 4.3 to deduce

$$
\begin{aligned}
\mathrm{II} &\le \fint_{B_\rho^+(x_0)} \big|a(\hat{x},0,Du)-a(x,0,Du)\big|\,dx \\
&\le c\fint_{B_\rho^+(x_0)} \omega\big(|x-\hat{x}|\big)\big(1+|Du|\big)^{p_2-1}\Big[1+\log(1+|Du|)\Big]\,dx \\
&\le c\omega(2\rho)\fint_{B_\rho^+(x_0)} \big(1+|Du|\big)^{p_2}\,dx \\
&\le c(M)\rho^\alpha.
\end{aligned}
$$

To estimate III, we use **(H5)**, the bound $\big|(D_nu)_{x_0,\rho}\,x_n\big| < M\rho$, keeping in mind Corollary 4.3 (i)

$$
\begin{aligned}
\mathrm{III} &\le \fint_{B_\rho^+(x_0)} \big|a(x,0,Du)-a(x,(D_nu)^+_{x_0,\rho}\,x_n,Du)\big|\,dx \\
&\le c\fint_{B_\rho^+(x_0)} \omega_\xi\big(|\,(D_nu)^+_{x_0,\rho}\,x_n|\big)\big(1+|Du|\big)^{p(x)-1}\,dx \\
&\le c\omega_\xi(M\rho)\fint_{B_\rho^+(x_0)} \big(1+|Du|\big)^{p_2-1}\,dx \\
&\le c\rho^\alpha.
\end{aligned}
$$

Owing to **(H5)**, Hölder's inequality, the bound $\omega_\xi \le 1$ and Corollary 4.2, then applying Jensen and Poincaré's inequalities, while keeping in mind $|\,(D_nu)^+_{x_0,\rho}\otimes e_n| \le M$ and

Corollary 4.3, we have for IV that

$$
\begin{aligned}
\mathrm{IV} &\leq \fint_{B_\rho^+(x_0)} \left| a(x, (D_n u)^+_{x_0,\rho}\, x_n, Du) - a(x, u, Du) \right| dx \\
&\leq c \fint_{B_\rho^+(x_0)} \omega_\xi\big(|u - (D_n u)^+_{x_0,\rho}\, x_n|\big)\big(1 + |Du|^2\big)^{\frac{p_2-1}{2}}\, dx \\
&\leq c\left(\fint_{B_\rho^+(x_0)} \omega_\xi^{p_2}\big(|u - (D_n u)^+_{x_0,\rho}\, x_n|\big)\, dx\right)^{\frac{1}{p_2}} \left(\fint_{B_\rho^+(x_0)} \big(1 + |Du|\big)^{p_2}\, dx\right)^{\frac{p_2-1}{p_2}} \\
&\leq c\left(\fint_{B_\rho^+(x_0)} \omega_\xi\big(|u - (D_n u)^+_{x_0,\rho}\, x_n|\big)\, dx\right)^{\frac{1}{p_2}} \\
&\leq c\omega_\xi^{\frac{1}{p_2}}\left(\fint_{B_\rho^+(x_0)} |u - (D_n u)^+_{x_0,\rho}\, x_n|\, dx\right) \\
&\leq c\omega_\xi^{\frac{1}{p_2}}\left(C\rho \fint_{B_\rho^+(x_0)} |Du - (D_n u)^+_{x_0,\rho} \otimes e_n|\, dx\right) \\
&\leq c\omega_\xi^{\frac{1}{p_2}}\left(C\rho \fint_{B_\rho^+(x_0)} M + |Du|\, dx\right) \\
&\leq c\omega_\xi^{\frac{1}{p_2}}\left(C\rho \fint_{B_\rho^+(x_0)} 1 + |Du|^{p_2}\, dx\right) \\
&\leq c\omega_\xi^{\frac{1}{p_2}}(C\rho) \\
&\leq c\rho^{\frac{\alpha}{p_2}}.
\end{aligned}
$$

When the inhomogeneity b satisfies the controllable growth condition **(B1)**, we can estimate by Corollary 4.3

$$
\begin{aligned}
\mathrm{V} &\leq \fint_{B_\rho^+(x_0)} |b(x, u, Du) \cdot \varphi|\; dx \\
&\leq L \fint_{B_\rho^+(x_0)} \big(1 + |Du|^2\big)^{\frac{p(x)-1}{2}} \rho\, dx \\
&\leq \rho c \fint_{B_\rho^+(x_0)} (1 + |Du|)^{p_2 - 1}\; dx \\
&\leq c\rho^\beta.
\end{aligned}
$$

Note that when the inhomogeneity satisfies the natural growth condition **(B2)** the estimates are analogous.

Assembling our terms we have in either case

$$
\begin{aligned}
&\left| \fint_{B_\rho(x_0)} D_z a(x_0, (u)^+_{x_0,\rho}, \Lambda)(Du - \Lambda) \cdot D\varphi\, dx \right| \\
&\qquad\qquad \leq c_1\Big(\mu_M\big(\mathcal{C}^{\frac{1}{2}}(x_0, \Lambda, \rho)\big)\mathcal{C}^{\frac{1}{2}}(x_0, \Lambda, \rho) + \mathcal{C}(x_0, \Lambda, \rho) + \rho^\beta\Big).
\end{aligned}
$$

This shows the claim for test functions satisfying $\|D\varphi\|_{L^\infty(B_\rho(x_0),\mathbb{R}^N)} = 1$. For general φ we simply rescale the function, testing instead with $\psi = \frac{\varphi}{\|D\varphi\|}$ to attain the full result. $\square$

Remark 6.8. *If Du is a priori bounded, then u is Lipschitz continuous and we easily have via Corollary 4.3 (i),*

$$\begin{aligned}
\mathrm{III} &\leq c \fint_{B_\rho^+(x_0)} \omega_\xi\big(|u - (D_n u)^+_{x_0,\rho}\, x_n|\big)\big(1 + |Du|^2\big)^{\frac{p(x)-1}{2}}\, dx \\
&\leq c \fint_{B_\rho^+(x_0)} \omega_\xi(C\rho)(1 + |Du|)^{p_2 - 1}\, dx \\
&\leq c\omega_\xi(C\rho) \\
&\leq c\rho^\alpha,
\end{aligned}$$

and our new estimate is satisfied with $\beta = \alpha$. The analogous estimate of course holds in the interior.

Note that we can immediately put this estimate into a more convenient form, as in [Bec11b]. Writing $\delta_0 \in (0,1)$ to be fixed later, we may write

$$\begin{aligned}
&c_1\Big(\mu_M\big(\mathcal{C}^{\frac{1}{2}}(x_0,\Lambda,\rho)\big)\mathcal{C}^{\frac{1}{2}}(x_0,\Lambda,\rho) + \mathcal{C}(x_0,\Lambda,\rho) + \rho^\beta\Big) \\
&\qquad\leq \sqrt{c_1^2\mathcal{C}(x_0,\Lambda,\rho) + \frac{2c_1^2}{\delta_0^2}\rho^{2\beta}}\sqrt{\Big(\mu_M\big(\mathcal{C}^{\frac{1}{2}}(x_0,\Lambda,\rho)\big) + \mathcal{C}(x_0,\Lambda,\rho)\Big)^2 + \frac{\delta_0^2}{2}}.
\end{aligned}$$

Application of the $\mathcal{A}$-harmonic approximation lemma

Having established this preliminary estimate, we find ourselves in a position where fixing these excesses small enough will allow us to invoke the $\mathcal{A}$-harmonic approximation lemma. The a priori bounds on the solution to the linearised PDE enable us to combine estimates of its linearisation with our Caccioppoli inequality in order to demonstrate a preliminary rescaling estimate on the Camponato style excess functional (3.11). This estimate is then iterated and an interpolation argument is provided to reproduce the estimate at all scales.

Preliminary Smallness Estimates

We now further restrict ρ_0 if necessary, such that for all $0 < \rho \leq \rho_0$ there holds

$$\kappa := \sqrt{c_1^2\mathcal{C}(x_0,\Lambda,\rho) + \frac{2c_1^2}{\delta_0^2}\rho^{2\beta}} \leq 1, \tag{6.8}$$

and

$$\Big(\mu_M\big(\mathcal{C}^{\frac{1}{2}}(x_0,\Lambda,\rho)\big)+\mathcal{C}(x_0,\Lambda,\rho)\Big)^2\leq\frac{\delta_0^2}{2}.$$

$\mathcal{A}$-harmonic approximation

Fixing $\Lambda=(D_nu)^+_{x_0,\rho}\otimes e_n$ for boundary points x_0, we have now satisfied the conditions of Lemma 3.12 with

$$\mathcal{A}:=D_za\big(\hat{x},0,(D_nu)^+_{x_0,\rho}\otimes e_n\big)$$

and

$$w:=u-(D_nu)^+_{x_0,\rho}\,x_n.$$

For interior points we set $\Lambda=(Du)_{x_0,\rho}$ so Lemma 3.12 is satisfied with

$$\mathcal{A}:=D_za\big(\hat{x},(u)_{x_0,\rho},(Du)_{x_0,\rho}\big)$$

and

$$w:=u-(u)_{x_0,\rho}-(Du)_{x_0,\rho}(x-x_0).$$

This implies the existence of an $\mathcal{A}$-harmonic function h satisfying the a priori estimates

$$\sup_{B_\rho^+(x_0)}\big(|Dh|+\rho|D^2h|\big)\leq c_h,$$

and

$$\fint_{B_\rho^+(x_0)}\left|V\left(\frac{w-\kappa h}{\rho}\right)\right|^2dx\leq\kappa^2\varepsilon.$$

Preliminary decay estimate

Lemma 6.9. *Fix $\sigma=\min\left\{\frac{\alpha}{2},\frac{\alpha}{p_2}\right\}$ and $\rho<\rho_0$ from Lemma 6.2. For each $M>2$ and $\sigma<\tilde{\beta}<1$, there exists constants $\theta,\hat{\varepsilon}\in(0,1)$ with dependencies*

$$\theta=\theta(n,N,p_2,\gamma_1,\gamma_2,L/\nu,L_1,L_2,E,\omega_p,M,\tilde{\beta},\alpha),$$
$$\hat{\varepsilon}=\hat{\varepsilon}(n,N,p_2,\gamma_1,\gamma_2,L/\nu,L_1,L_2,E,\omega_p,M,\mu_M,\tilde{\beta},\alpha),$$

for which the following holds: If $u\in W^{1,p(\,\cdot\,)}\big(\Omega,\mathbb{R}^N\big)$ is a weak solution to (6.1) *under assumptions* **(H1)**–**(H6)**, *and either* **(B1)**, *or* **(B2)** *under the additional assumption* *that the solution is bounded with $2L_1\|u\|_{L^\infty} < \nu$, and x_0 satisfies $B_\rho(x_0) \subset\subset \Omega$ with the smallness conditions*

$$\rho + \mathcal{C}(x_0,\rho) < \hat{\varepsilon}, \qquad \textit{and} \qquad |\,(Du)_{x_0,\rho}\,| + |\,(u)_{x_0,\rho}\,| \le M \tag{6.9}$$

hold, then we have then we have

$$\mathcal{C}(x_0,\theta\rho) \le \theta^{2\tilde{\beta}}\mathcal{C}(x_0,\rho) + c_*\rho^{2\sigma}. \tag{6.10}$$

Here the constant c_* depends on $n, N, L/\nu, L_1, L_2, E, \omega_p, M, p_2, \gamma_1, \gamma_2$ and $\tilde{\beta}$.

Lemma 6.10 (Boundary Decay Estimate)**.** *Fix $\sigma = \min\left\{\frac{\alpha}{2}, \frac{\alpha}{p_2}\right\}$ and $\rho < \rho_0$ from Lemma 6.3. For each $M > 2$ and $\sigma < \tilde{\beta} < 1$, there exists constants $\theta, \hat{\varepsilon} \in (0,1)$ with dependencies*

$$\begin{aligned}\theta &= \theta(n, N, p_2, \gamma_1, \gamma_2, L/\nu, L_1, L_2, E, \omega_p, M, \tilde{\beta}, \alpha),\\ \hat{\varepsilon} &= \hat{\varepsilon}(n, N, p_2, \gamma_1, \gamma_2, L/\nu, L_1, L_2, E, \omega_p, M, \mu_M, \tilde{\beta}, \alpha),\end{aligned}$$

for which the following holds: If $u \in W^{1,p(\,\cdot\,)}\big(B^+, \mathbb{R}^N\big)$ is a solution to the transformed system (6.2) *under assumptions* **(H1)**–**(H6)***, and either* **(B1)***, or* **(B2)** *under the additional assumption that the solution is bounded with $2L_1\|u\|_{L^\infty} < \nu$, then for each $x_0 \in \Gamma$, $\rho < \min\{\rho_0, 1 - |x_0|\}$ satisfying the smallness conditions*

$$\rho + \mathcal{C}(x_0,\rho) < \hat{\varepsilon}, \qquad \textit{and} \qquad |\,(D_n u)^+_{x_0,\rho} \otimes e_n| \le M, \tag{6.11}$$

we have

$$\mathcal{C}(x_0,\theta\rho) \le \theta^{2\tilde{\beta}}\mathcal{C}(x_0,\rho) + c_*\rho^{2\sigma}. \tag{6.12}$$

Here the constant c_* depends only on $n, N, p_2, L/\nu, L_1, L_2, E, \omega_p, M, \gamma_1, \gamma_2$ and $\tilde{\beta}$.

Proof of Lemma 6.10: We again perform the calculations for boundary points, with obvious modifications being required for those in the interior. Fixing $\theta \in (0, \frac{1}{8})$, we again write $\hat{p} = \max\{2, p_2\}$ and fix $\varepsilon = \theta^{n+\hat{p}+2}$. We define the affine map

$$\xi_\rho(x) := \big(\,(D_n u)^+_{x_0,\rho} + \kappa D_n h(x_0)\big)x_n \in \mathbb{R}^N, \tag{6.13}$$

where the function h is taken from Lemma 3.12. Noting $h(x_0) = 0$, Taylor's theorem gives

$$|h(x) - D_n h(x_0)x_n| \le |\mathcal{R}(x)| \le \sup_{B^+_\rho(x_0)} |D^2 h(x)|\rho^2, \tag{6.14}$$

for $x \in B^+_\rho(x_0)$.

We can use Lemma 3.4 (ii) and (iv) and Lemma 3.12, together with (6.13) and (6.14) to compute

$$
\begin{aligned}
&\fint_{B^+_{\theta\rho}(x_0)} \left| V\left(\frac{u - \xi_{2\theta\rho}(x)}{2\theta\rho} \right) \right|^2 dx \qquad (6.15)\\
&\quad = \fint_{B^+_{\theta\rho}(x_0)} \left| V\left(\frac{w - \kappa D_n h(x_0) x_n}{2\theta\rho} \right) \right|^2 dx \\
&\quad \leq c \fint_{B^+_{\theta\rho}(x_0)} \left| V\left(\frac{w - \kappa h}{2\theta\rho} \right) \right|^2 dx + c \fint_{B^+_{\theta\rho}(x_0)} \left| V\left(\frac{\kappa\big(h - D_n h(x_0) x_n\big)}{2\theta\rho} \right) \right|^2 dx \\
&\quad \leq c\theta^{-n} \fint_{B^+_{\frac{\rho}{4}}(x_0)} \left| V\left(\frac{w - \kappa h}{2\theta\rho} \right) \right|^2 dx \\
&\qquad + c \fint_{B^+_{\theta\rho}(x_0)} \left| \frac{\kappa\big(h - Dh(x_0) x_n\big)}{2\theta\rho} \right|^2 + \left| \frac{\kappa\big(h - Dh(x_0) x_n\big)}{2\theta\rho} \right|^{\hat{p}} dx \\
&\quad \leq c\theta^{-n-\hat{p}} \fint_{B^+_{\frac{\rho}{4}}(x_0)} \left| V\left(\frac{w - \kappa h}{\rho/2} \right) \right|^2 dx \\
&\qquad + \fint_{B^+_{\theta\rho}(x_0)} \sup_{y \in B^+_\rho(x_0)} \left| \frac{\kappa |D^2 h(y)| (2\theta\rho)^2}{2\theta\rho} \right|^2 + \sup_{y \in B^+_\rho(x_0)} \left| \frac{\kappa |D^2 h(y)| (2\theta\rho)^2}{2\theta\rho} \right|^{\hat{p}} dx \\
&\quad \leq c\theta^{-n-\max\{2,p_2\}} \fint_{B^+_{\frac{\rho}{4}}(x_0)} \left| V\left(\frac{w - \kappa h}{\rho/2} \right) \right|^2 dx \\
&\qquad + \fint_{B^+_{\theta\rho}(x_0)} \left| \frac{\kappa c_h (2\theta\rho)^2}{2\theta\rho^2} \right|^2 + \left| \frac{\kappa c_h (2\theta\rho)^2}{2\theta\rho^2} \right|^{\hat{p}} dx \\
&\quad \leq c\theta^{-n-\max\{2,p_2\}} \fint_{B^+_{\frac{\rho}{4}}(x_0)} \left| V\left(\frac{w - \kappa h}{\rho/2} \right) \right|^2 dx + c\big(\theta^2\kappa^2 + (\kappa\theta)^{\hat{p}}\big) \\
&\quad \leq c(n, N, p_2, \nu, L)\theta^2\kappa^2.
\end{aligned}
$$

Here we have used our choice of ε, κ and $\theta < 1$.

In the final step we begin by sharpening condition (6.8) to

$$
\kappa \leq \frac{1}{c_h}. \tag{6.16}
$$

This ensures via (3.16) and (6.9) that

$$
|D\xi_\rho| = \big|\, (D_n u)^+_{x_0,\rho} \otimes e_n + \kappa Dh(x_0) \otimes e_n \big| \leq M + 1,
$$

which is required for Lemma 6.3 and Lemma 6.7.

We can now exploit Lemma 3.8, keeping in mind Lemma 6.3 with Remark 6.4 and

(6.15), to estimate

$$\begin{aligned}
\mathcal{C}(x_0,\theta\rho) &= \fint_{B^+_{\theta\rho}(x_0)} \left|V(Du-(D_nu)^+_{x_0,\theta\rho}\otimes e_n)\right|^2\,dx \\
&\leq \fint_{B^+_{\theta\rho}(x_0)} |V(Du-D\xi_{2\theta\rho})|^2\,dx \\
&\leq c\fint_{B^+_{\theta\rho}(x_0)} \left|V\left(\frac{u-\xi_{2\theta\rho}(x)}{2\theta\rho}\right)\right|^2\,dx + c\rho^\alpha \\
&\leq c\left(\theta^2\kappa^2\right) + c\rho^\alpha \\
&\leq c\left(\theta^2\mathcal{C}(x_0,\rho)+\delta^{-2}\rho^{2\beta}\right) + c\rho^\alpha \\
&\leq c\theta^2\mathcal{C}(x_0,\rho) + c\rho^{\min\left\{\alpha,\frac{2\alpha}{p_2}\right\}} \\
&\leq c\theta^2\mathcal{C}(x_0,\rho) + c\rho^{2\sigma}.
\end{aligned}$$

For fixed $\tilde\beta\in(\sigma,1)$, put $\theta\in(0,\frac{1}{8})$ small enough to ensure $c\theta^2\leq\theta^{2\tilde\beta}$. The result then follows. $\square$

Remark 6.11. *If we know that Du is bounded, and hence u is Lipschitz continuous, we can use Remark 6.5 together with Remark 6.8, fixing $\beta=2\alpha$. The estimate now improves to*

$$\begin{aligned}
\mathcal{C}(x_0,\theta\rho) &\leq \fint_{B^+_{\theta\rho}(x_0)} \left|V(Du-\xi_{2\theta\rho}(x))\right|^2 dx \\
&\leq c\fint_{B^+_{\theta\rho}(x_0)} \left|V\left(\frac{u-\xi_{2\theta\rho}(x)}{2\theta\rho}\right)\right|^2 dx + c\rho^{2\alpha} \\
&\leq c\big(\theta^2\kappa^2\big) + c\rho^{2\alpha} \\
&\leq c\big(\theta^2\mathcal{C}(x_0,\rho)+\delta^{-2}\rho^{2\beta}\big) + c\rho^{2\alpha} \\
&\leq c\theta^2\mathcal{C}(x_0,\rho) + c\rho^{2\alpha},
\end{aligned}$$

and (6.12) *holds with $\sigma=\alpha$, for appropriate choice of $\tilde\beta$ and θ.*

Excess decay iteration

We can now iterate this procedure to show the following:

Lemma 6.12. *For every $M>2$ and $\sigma<\tilde\beta<1$, there exists some constant $0<\bar\varepsilon<1$ such that whenever the smallness conditions*

$$\rho+\mathcal{C}(x_0,\rho)<\bar\varepsilon, \qquad \textit{and} \qquad \left|(Du)_{x_0,\rho}\right|+\left|(u)_{x_0,\rho}\right|\leq\frac{M}{2} \tag{6.17}$$

hold for interior points, or

$$\rho + \mathcal{C}(x_0,\rho) < \bar{\varepsilon}, \qquad \textit{and} \qquad \big|\,(D_n u)^+_{x_0,\rho} \otimes e_n\big| \leq \frac{M}{2} \tag{6.18}$$

hold for transformed boundary points, then for all $0 < r \leq \rho$ *we have*

$$\mathcal{C}(x_0,r) \leq c_d \left[\left(\frac{r}{\rho}\right)^{2\tilde{\beta}} \mathcal{C}(x_0,\rho) + \rho^{2\sigma} \right].$$

Here, c_d *has the same dependencies as* c_**, and* $\bar{\varepsilon}$ *as* $\hat{\varepsilon}$*.*

Proof of Lemma 6.12: We again show the boundary case, with the interior version being clear from obvious modifications. We shall begin by using the same θ and $\hat{\varepsilon}$, now imposing the stronger smallness condition on ρ:

$$\rho + \frac{c_*}{\theta^{2\sigma} - \theta^{2\tilde{\beta}}} \rho^{2\sigma} + \mathcal{C}(x_0,\rho) < \hat{\varepsilon}. \tag{6.19}$$

We will show analogues of Lemma 6.9 for radii that are integer powers of θ, then interpolate between them to attain the full result.

We proceed to calculate directly for any fixed $j \in \mathbb{N}$ that there holds via Lemma 6.10

$$\mathcal{C}(x_0,\theta^j\rho) \leq \theta^{2\tilde{\beta}j}\mathcal{C}(x_0,\rho) + \frac{c_*}{\theta^{2\sigma} - \theta^{2\tilde{\beta}}}\big(\theta^j\rho\big)^{2\sigma}, \tag{6.20}$$

$$\theta^j\rho + \mathcal{C}(x_0,\theta^j\rho) \leq \hat{\varepsilon}, \tag{6.21}$$

$$\big|(D_n u)^+_{x_0,\theta^j\rho} \otimes e_n\big| \leq M. \tag{6.22}$$

To show (6.20) we can use (6.12) j times to attain the geometric series

$$\begin{aligned}
\mathcal{C}\big(x_0,\theta^j\rho\big) &\leq \theta^{2\tilde{\beta}}\mathcal{C}\big(x_0,\theta^{j-1}\rho\big) + c_*\big(\theta^{j-1}\rho\big)^{2\sigma} \\
&\leq \theta^{2\tilde{\beta}}\Big(\theta^{2\tilde{\beta}}\mathcal{C}\big(x_0,\theta^{j-2}\rho\big) + c_*\big(\theta^{j-2}\rho\big)^{2\sigma}\Big) + c_*\big(\theta^{j-1}\rho\big)^{2\sigma} \\
&= \theta^{(2\tilde{\beta})2}\mathcal{C}(x_0,\theta^{j-2}\rho) + c_*\Big(\theta^{2\tilde{\beta}}\big(\theta^{j-2}\rho\big)^{2\sigma} + \big(\theta^{j-1}\rho\big)^{2\sigma}\Big) \\
&\leq \theta^{2\tilde{\beta}j}\mathcal{C}(x_0,\rho) + c_*\sum_{k=0}^{j-1}\rho^{2\sigma}\theta^{2\tilde{\beta}k}\theta^{2\sigma(j-1-k)} \\
&= \theta^{2\tilde{\beta}j}\mathcal{C}(x_0,\rho) + c_*\sum_{k=0}^{j-1}\rho^{2\sigma}\theta^{2(\tilde{\beta}-\sigma)k}\theta^{2\sigma(j-1)} \\
&= \theta^{2\tilde{\beta}j}\mathcal{C}(x_0,\rho) + c_*\theta^{-2\sigma}\big(\theta^j\rho\big)^{2\sigma}\sum_{k=0}^{j-1}\theta^{2(\tilde{\beta}-\sigma)k} \\
&\leq \theta^{2\tilde{\beta}j}\mathcal{C}(x_0,\rho) + \frac{c_*\big(\theta^j\rho\big)^{2\sigma}}{\theta^{2\sigma} - \theta^{2\tilde{\beta}}}.
\end{aligned}$$

This shows (6.20). We are now able to calculate (6.21) under assumption (6.19), since

$$\begin{aligned}\theta^j\rho+\mathcal{C}(x_0,\theta^j\rho) &\le \theta^j\rho+\theta^{2\tilde{\beta}j}\mathcal{C}(x_0,\rho)+\frac{c_*}{\theta^{2\sigma}-\theta^{2\tilde{\beta}}}\big(\theta^j\rho\big)^{2\sigma}\\ &\le \rho+\mathcal{C}(x_0,\rho)+\frac{c_*}{\theta^{2\sigma}-\theta^{2\tilde{\beta}}}\rho^{2\sigma}\\ &\le \hat{\varepsilon}.\end{aligned}$$

When showing (6.22), for ease of notation we write $(D_nu)^+_{x_0,\rho}$ for $(D_nu)^+_{x_0,\rho}\otimes e_n$. When $p_2\ge 2$, we use elementary integration and Lemma 3.4 (iv), then Hölder's inequality, keeping in mind (6.20) and the choice of $0<\sigma<\tilde{\beta}<1$, followed by the concavity of $x\mapsto\sqrt{x}$ to calculate

$$\begin{aligned}\big|(D_nu)^+_{x_0,\theta^j\rho}-(D_nu)^+_{x_0,\rho}\big| &\le \Big|\sum_{k=1}^{j}(D_nu)^+_{x_0,\theta^k\rho}-(D_nu)^+_{x_0,\theta^{k-1}\rho}\Big|\\ &=\Big|\sum_{k=1}^{j}\fint_{B^+_{\theta^k\rho}(x_0)}D_nu-(D_nu)^+_{x_0,\theta^{k-1}\rho}\,dx\Big|\\ &\le\sum_{k=1}^{j}\fint_{B^+_{\theta^k\rho}(x_0)}\Big|D_nu-(D_nu)^+_{x_0,\theta^{k-1}\rho}\Big|\,dx\\ &=\sum_{k=0}^{j-1}\fint_{B^+_{\theta^{k+1}\rho}(x_0)}\Big|D_nu-(D_nu)^+_{x_0,\theta^{k}\rho}\Big|\,dx\\ &\le\sum_{k=0}^{j-1}\theta^{-n}\fint_{B^+_{\theta^{k}\rho}(x_0)}\Big|Du-(D_nu)^+_{x_0,\theta^{k}\rho}\Big|\,dx\\ &\le\sum_{k=0}^{j-1}\theta^{-n}\fint_{B^+_{\theta^{k}\rho}(x_0)}\Big|V\big(Du-(D_nu)^+_{x_0,\theta^{k}\rho}\big)\Big|\,dx\\ &\le\theta^{-n}\sum_{k=0}^{j-1}\mathcal{C}^{\frac12}(x_0,\theta^j\rho)\\ &\le\theta^{-n}\sum_{k=0}^{j-1}\sqrt{\theta^{2\tilde{\beta}j}\mathcal{C}(x_0,\rho)+\frac{c_*}{\theta^{2\sigma}-\theta^{2\tilde{\beta}}}\big(\theta^j\rho\big)^{2\sigma}}\\ &\le\theta^{-n}\sqrt{\mathcal{C}(x_0,\rho)+\frac{c_*}{\theta^{2\sigma}-\theta^{2\tilde{\beta}}}\rho^{2\sigma}}\sum_{k=0}^{j-1}\theta^{\sigma j}\\ &\le\frac{\theta^{-n}}{1-\theta^{\sigma}}\sqrt{\mathcal{C}(x_0,\rho)+\frac{c_*}{\theta^{2\sigma}-\theta^{2\tilde{\beta}}}\rho^{2\sigma}}\\ &\le\frac{\theta^{-n}}{1-\theta^{\sigma}}\left(\sqrt{\mathcal{C}(x_0,\rho)}+\sqrt{\frac{c_*}{\theta^{2\sigma}-\theta^{2\tilde{\beta}}}\rho^{2\sigma}}\right)\\ &\le\frac{\theta^{-n}}{1-\theta^{\sigma}}\left(\mathcal{C}^{\frac12}(x_0,\rho)+\frac{c_*}{\theta^{2\sigma}-\theta^{2\tilde{\beta}}}\rho^{\sigma}\right).\end{aligned}$$

Here we have assumed without losing any generality that $c_* > 1$.

On the other hand, when $1 < p_2 < 2$ we repeat these calculations, noting the difference in Lemma 3.4 (iv) to find

$$\begin{aligned}
\left|(D_nu)^+_{x_0,\theta^j\rho} - (D_nu)^+_{x_0,\rho}\right| &\leq \left|\sum_{k=1}^{j}(D_nu)^+_{x_0,\theta^k\rho} - (D_nu)^+_{x_0,\theta^{k-1}\rho}\right| \\
&\leq \sum_{k=0}^{j-1} \fint_{B_{\theta^{k+1}\rho}(x_0)} \theta^{-n}\left|Du - (D_nu)^+_{x_0,\theta^k\rho}\otimes e_n\right| dx \\
&\leq \theta^{-n}\sum_{k=0}^{j-1} \mathcal{C}^{\frac{1}{2}}(x_0,\theta^j\rho) + \mathcal{C}(x_0,\theta^j\rho)^{\frac{1}{p_2}}.
\end{aligned}$$

The first term in the sum has already been estimated, and using the fact that $x \mapsto x^{\frac{1}{p_2}}$ is concave, we can similarly handle the latter term

$$\begin{aligned}
\theta^{-n}\sum_{k=0}^{j-1}\mathcal{C}(x_0,\theta^j\rho)^{\frac{1}{p_2}} &\leq \theta^{-n}\sum_{k=0}^{j-1}\left(\theta^{2\tilde{\beta}j}\mathcal{C}(x_0,\rho) + \frac{c_*}{\theta^{2\sigma}-\theta^{2\tilde{\beta}}}\left(\theta^j\rho\right)^{2\sigma}\right)^{\frac{1}{p_2}} \\
&\leq \theta^{-n}\left(\mathcal{C}(x_0,\rho) + \frac{c_*}{\theta^{2\sigma}-\theta^{2\tilde{\beta}}}\rho^{2\sigma}\right)^{\frac{1}{p_2}}\sum_{k=0}^{j-1}\theta^{\frac{2\sigma j}{p_2}} \\
&\leq \theta^{-n}\left(\mathcal{C}(x_0,\rho)^{\frac{1}{p_2}} + \left(\frac{c_*}{\theta^{2\sigma}-\theta^{2\tilde{\beta}}}\rho^{2\sigma}\right)^{\frac{1}{p_2}}\right)\sum_{k=0}^{j-1}\theta^{\sigma j} \\
&\leq \frac{\theta^{-n}}{1-\theta^{\sigma}}\left(\mathcal{C}^{\frac{1}{2}}(x_0,\rho) + \left(\frac{c_*}{\theta^{2\sigma}-\theta^{2\tilde{\beta}}}\right)^{\frac{1}{p_2}}\rho^{\frac{2\sigma}{p_2}}\right) \\
&\leq \frac{\theta^{-n}}{1-\theta^{\sigma}}\left(\mathcal{C}^{\frac{1}{2}}(x_0,\rho) + \frac{c_*}{\theta^{2\sigma}-\theta^{2\tilde{\beta}}}\rho^{\sigma}\right).
\end{aligned}$$

In either case, imposing the smallness condition

$$\frac{\theta^{-\frac{n}{2}}}{1-\theta^{\sigma}}\left(\mathcal{C}^{\frac{1}{2}}(x_0,\rho) + \frac{c_*}{\theta^{2\sigma}-\theta^{2\tilde{\beta}}}\rho^{\sigma}\right) \leq \frac{M}{4} \tag{6.23}$$

gives the result.

We now note that for each $0 < r \leq \rho$ we have $\theta^j\rho < r \leq \theta^{j-1}\rho$ for some $j \in \mathbb{N}$. We can now use (6.20) and Lemma 3.8 to show

$$
\begin{aligned}
\mathcal{C}(x_0,r) &= \fint_{B_r^+(x_0)} \left|V\big(Du-(D_nu)^+_{x_0,r}\big)\right|^2 dx \\
&\le \fint_{B_r^+(x_0)} \left|V\big(Du-(D_nu)^+_{x_0,\theta^{j-1}\rho}\big)\right|^2 dx \\
&\le \frac{1}{\theta^n} \fint_{B^+_{\theta^{j-1}\rho}(x_0)} \left|V\big(Du-(D_nu)^+_{x_0,\theta^{j-1}\rho}\big)\right|^2 dx \\
&= \frac{1}{\theta^n}\mathcal{C}(x_0,\theta^{j-1}\rho) \\
&\le \frac{c}{\theta^n}\left[\theta^{2\tilde\beta(j-1)}\mathcal{C}(x_0,\rho)+(\theta^{j-1}\rho)^{2\sigma}\right] \\
&\le \frac{c}{\theta^{n+2\tilde\beta}}\left[\theta^{2\tilde\beta j}\mathcal{C}(x_0,\rho)+(\theta^{j}\rho)^{2\sigma}\right] \\
&\le \frac{c}{\theta^{n+2\tilde\beta}}\left[\left(\frac{r}{\rho}\right)^{2\tilde\beta}\mathcal{C}(x_0,\rho)+r^{2\sigma}\right].
\end{aligned}
$$

Choosing $\bar\varepsilon$ small enough to ensure the smallness conditions (6.19) and (6.23) are implied by (6.18), the proof is complete. □

Proof of Theorem 6.1

We now prove the assertion of Theorem 6.1, the Hölder continuity of Du for weak solutions to (6.1). We will do this in multiple steps, with the first proposition dealing with interior points, and the second taking a number of different cases for points on or near the boundary in the model situation. We then combine these facts with arguments from Chapter 5, which allows us to conclude Theorem 6.1.

We begin by demonstrating via an interpolation argument that, up to modifying the values in our smallness assumptions, we can equivalently consider the L^1, L^{p_2} and $L^{p(x)}$-Lebesgue points of Du.

We will again write $(D_nu)^+_{x_0,\rho}$ for $(D_nu)^+_{x_0,\rho}\otimes e_n$. Observe that under the assumptions of Lemma 6.12, Corollary 4.3 (i) and (ii) give

$$
\begin{aligned}
\fint_{B_\rho^+(x_0)} \left|Du-(D_nu)^+_{x_0,\rho}\right| dx &\le \fint_{B_\rho^+(x_0)} M+|Du|\,dx \\
&\le C\big(1+\left|(D_nu)^+_{x_0,\rho}\right|\big) \\
&\le C.
\end{aligned}
$$

Furthermore, under the same assumptions together with Corollary 4.3 (i) and (ii),

$$\begin{aligned}\fint_{B_\rho^+(x_0)} |Du-(D_nu)^+_{x_0,\rho}|^{p_2(1+\frac{\delta}{4})}\,dx &\le c\big(1+\big|\,(D_nu)^+_{x_0,\rho}\,\big|\big)^{p_2(1+\frac{\delta}{4})}\\ &\le C.\end{aligned}$$

In the spirit of [BDHS12], we use Lemma 4.4 with $s=p_2$, $p=1$ and $q=p_2(1+\frac{\delta}{4})$, which fixes $\theta=\frac{\delta}{p_2(4+\delta)-4}\in(0,1)$ to find

$$\|Du-(D_nu)^+_{x_0,\rho}\|_{L^{p_2}}\le \|Du-(D_nu)^+_{x_0,\rho}\|^{\theta}_{L^1}\|Du-(D_nu)^+_{x_0,\rho}\|^{1-\theta}_{L^{p_2(1+\frac{\delta}{4})}}.$$

Taking averages gives

$$\begin{aligned}\fint_{B_\rho^+(x_0)}\big|Du-(D_nu)^+_{x_0,\rho}\big|^{p_2}\,dx &\le \bigg(\fint_{B_\rho^+(x_0)}\big|Du-(D_nu)^+_{x_0,\rho}\,\big|\,dx\bigg)^{\frac{p_2\delta}{p_2(4+\delta)-4}}\\ &\qquad\times\bigg(\fint_{B_\rho^+(x_0)}\big|Du-(D_nu)^+_{x_0,\rho}\,\big|^{p_2(1+\frac{\delta}{4})}\,dx\bigg)^{\frac{4p_2-4}{p_2(4+\delta)-4}}\\ &\le C\bigg(\fint_{B_\rho^+(x_0)}\big|Du-(D_nu)^+_{x_0,\rho}\,\big|\,dx\bigg)^{\frac{p_2\delta}{p_2(4+\delta)-4}},\end{aligned}\tag{6.24}$$

since $\frac{p_2\delta}{p_2(4+\delta)-4}+\frac{4p_2-4}{p_2(4+\delta)-4}=1$.

Next, under our smallness assumptions on ρ and fixed $x_0\in\Omega_0$, we note that when $1<p_2<2$ Lemma 3.4 (iv) and (6.24) imply

$$\begin{aligned}\mathcal{C}(x_0,\rho)&=\fint_{B_\rho^+(x_0)}\big|V\big(Du-(D_nu)^+_{x_0,\rho}\big)\big|^2\,dx\\ &\le\fint_{B_\rho^+(x_0)}\big|Du-(D_nu)^+_{x_0,\rho}\,\big|^{p_2}dx\\ &\le C\bigg(\fint_{B_\rho^+(x_0)}\big|Du-(D_nu)^+_{x_0,\rho}\,\big|\,dx\bigg)^{\frac{p_2\delta}{p_2(4+\delta)-4}}.\end{aligned}$$

When $p \geq 2$ we have by Lemma 3.4 (iv) and Hölder's inequality, followed by (6.24)

$$
\begin{aligned}
\mathcal{C}(x_0,\rho) &= \fint_{B_\rho^+(x_0)} \big|V\big(Du-(D_nu)^+_{x_0,\rho}\big)\big|^2\,dx \\
&\leq \fint_{B_\rho^+(x_0)} \big|Du-(D_nu)^+_{x_0,\rho}\big|^{p_2}\,dx + \fint_{B_\rho^+(x_0)} \big|Du-(D_nu)^+_{x_0,\rho}\big|^{2}\,dx \\
&\leq \fint_{B_\rho^+(x_0)} \big|Du-(D_nu)^+_{x_0,\rho}\big|^{p_2}\,dx + \bigg(\fint_{B_\rho^+(x_0)} \big|Du-(D_nu)^+_{x_0,\rho}\big|^{p_2}\,dx\bigg)^{\frac{2}{p_2}} \\
&\leq C\bigg(\fint_{B_\rho^+(x_0)} \big|Du-(D_nu)^+_{x_0,\rho}\big|\,dx\bigg)^{\frac{p_2\delta}{p_2(4+\delta)-4}} \\
&\qquad + C\bigg(\fint_{B_\rho^+(x_0)} \big|Du-(D_nu)^+_{x_0,\rho}\big|\,dx\bigg)^{\frac{2\delta}{p_2(4+\delta)-4}}.
\end{aligned}
$$

Clearly, the same estimates hold in the interior if we replace $B_\rho^+(x_0)$ by $B_\rho(x_0)$.

Setting

$$\Sigma_{1,\Gamma} := \bigg\{x_0 \in \Gamma : \liminf_{\rho\downarrow 0} \fint_{B_\rho^+(x_0)} \big|Du-(D_nu)^+_{x_0,\rho}\otimes e_n\big|\,dx > 0\bigg\}$$

and

$$\Sigma_{2,\Gamma} := \bigg\{x_0 \in \Gamma : \limsup_{\rho\downarrow 0} \big|\,(D_nu)^+_{x_0,\rho}\otimes e_n\big| = \infty\bigg\},$$

we will write $\Gamma_0 = \Gamma \setminus (\Sigma_{1,\Gamma}\cup\Sigma_{2,\Gamma})$. For the interior case we similarly put

$$\Sigma_{1,\Omega} := \bigg\{x_0 \in \Omega : \liminf_{\rho\downarrow 0} \fint_{B_\rho(x_0)} \big|Du-(Du)_{x_0,\rho}\big|\,dx > 0\bigg\}$$

and

$$\Sigma_{2,\Omega} := \bigg\{x_0 \in \Omega : \limsup_{\rho\downarrow 0} \big|\,(Du)_{x_0,\rho}\big| + \big|\,(u)_{x_0,\rho}\big| = \infty\bigg\},$$

and write $\Omega_0 = \Omega\setminus(\Sigma_{1,\Omega}\cup\Sigma_{1,\Omega})$. In either case, we have that for any $x_0\in\Gamma$ or $x_0\in\Omega_0$ there holds

$$\liminf_{\rho\downarrow 0}\mathcal{C}(x_0,\rho) = 0.$$

The interior setting

We first consider the case for interior points. We will begin by demonstrating rough Hölder continuity and obtain a fixed Hölder exponent depending on p_2. We then see that this result self-improves to give the full result.

Proposition 6.13. *Let $u \in W^{1,p(\,\cdot\,)}\big(\Omega, \mathbb{R}^N\big)$ be a weak solution to*

$$- \operatorname{div} a(x,u,Du) = b(x,u,Du) \quad \textit{in} \;\; \Omega,$$

under assumptions **(H1)**–**(H6)**, *where the inhomogeneity b satisfies either* **(B1)**, *or* **(B2)** *under the additional assumption that the solution is bounded with $2L_1 \|u\|_{L^\infty} < \nu$. Then the following hold:*

(i) $\operatorname{Reg}_{Du}(\Omega)$ *is relatively open in Ω,*

(ii) $u \in C^{1,\alpha}\big(\operatorname{Reg}_{Du}(\Omega), \mathbb{R}^N\big)$,

(iii) $\operatorname{Sing}_{Du}(\Omega) \subset \Sigma_{1,\Omega} \cup \Sigma_{2,\Omega}$.

In particular, we have $\mathcal{L}^n\big(\operatorname{Sing}_{Du}(\Omega)\big) = 0$.

Proof of Proposition 6.13: Taking a point $x_0 \in \Omega_0$, we infer the existence of some $M > 0$ such that

$$M := 1 + 4 \max\Big\{ \limsup_{\rho \downarrow 0} \big|(Du)_{x_0,\rho}\big|, \limsup_{\rho \downarrow 0} \big|(u)_{x_0,\rho}\big| \Big\},$$

and some $\rho_0 > \hat{\rho} > 0$ such that $B_{4\hat{\rho}}(x_0) \subset\subset \Omega$, with $\big|(Du)_{x_0,\hat{\rho}}\big|, \big|(u)_{x_0,\hat{\rho}}\big| \leq \frac{M}{2}$ and $\hat{\rho} + \mathcal{C}(x_0,\hat{\rho}) < \bar{\varepsilon}$ from Lemma 6.12. From the continuity of

$$x \mapsto \mathcal{C}(x,\rho), \quad \text{and} \quad x \mapsto (Du)_{x,\rho},$$

we have that (6.19) continues to hold for each point in a neighbourhood $B_R^+(x_0)$, for some $0 < R \leq \frac{\hat{\rho}}{2}$. We assume now without losing generality that $x_0 = 0$.

Recall that $\sigma = \min\{\frac{\alpha}{2}, \frac{\alpha}{p_2}\}$, and set θ small enough to fix $2\tilde{\beta} = 1 + \sigma$. We can invoke Lemma 6.12 to obtain

$$\mathcal{C}(x,r) \leq c\left[\left(\frac{r}{\rho_0}\right)^{1+\sigma} \mathcal{C}(x,\rho_0) + r^{2\sigma} \right] \tag{6.25}$$

for each $x \in B_R$ and $0 < r \leq \hat{\rho}$.

Now, when $p_2 \geq 2$ we have from Lemma 3.4 (*iv*), (6.25), and (6.19) that

$$\begin{aligned} r^{-2\sigma} \fint_{B_r(x)} \big|Du - (D_n u)_{x,r} \otimes e_n\big|^2 \, dx &\leq r^{-2\sigma} \mathcal{C}(x,r) \\ &\leq c\left[\frac{R^{1-\sigma}}{\rho_0^{1+\sigma}} \bar{\varepsilon} + 1 \right], \end{aligned} \tag{6.26}$$

for all $x \in B_R$ and $0 < r < 2R$.

This is the integral characterisation of Hölder continuity originally due to Campanato as Teorema I2 in [Cam63], see Theorem 2.1. Thus we deduce that the solution Du has a Hölder continuous representative and we obtain $u \in C^{1,\sigma}(B_R, \mathbb{R}^N)$.

Conversely, in the case where $1 < p < 2$, we use Lemma 3.4 (v), (6.25), and (6.19) to find

$$\begin{aligned} r^{-2\sigma} \fint_{B_r(x)} \left|V(Du) - \big(V(Du)\big)_{x,r}\right|^2 dx &\leq cr^{-2\sigma} \fint_{B_r(x)} \left|V(Du) - V\big((Du)_{x,r}\big)\right|^2 dx \\ &\leq cr^{-2\sigma}\mathcal{C}(x,r) \\ &\leq c\left[\frac{R^{1-\sigma}}{\rho_0^{1+\sigma}}\bar{\varepsilon} + 1\right], \end{aligned} \tag{6.27}$$

and so similarly, $V(Du)$ has a continuous representative whence $V(Du) \in C^{0,\sigma}(B_R, \mathbb{R}^N)$.

In particular, this implies $V(Du)$ is bounded on B_R, which in turn implies that Du is bounded via Lemma 3.4 (iv). This allows us use Lemma 3.4 (iii) and the Hölder continuity of $V(Du)$ to calculate that whenever $x, y \in B_R$ there holds

$$\begin{aligned} |Du(x) - Du(y)|^2 &\leq \left(1 + |Du(x)|^2 + |Du(y)|^2\right)^{\frac{2-p}{2}} \left|V(Du(x)) - V(Du(y))\right|^2 \\ &\leq C\left|V(Du(x)) - V(Du(y))\right|^2 \\ &\leq C|x-y|^{2\sigma}. \end{aligned} \tag{6.28}$$

We conclude that Du has a continuous representative with $u \in C^{1,\sigma}(B_R, \mathbb{R}^N)$.

Note that since Du is Hölder continuous, it is bounded on B_R. So u is Lipschitz continuous on this set, which allows us to use Remark 6.5 and Remark 6.8 in Remark 6.11. Iterating the above argument implies that $u \in C^{1,\alpha}(B_R, \mathbb{R}^N)$, where we have possibly restricted the value of R. At this stage, we remark that any dependence on p_2 is continuous, and so the constants remain bounded for finite values away from 1. They can therefore be replaced globally by constants depending on γ_1 and γ_2 in place of p_2. Furthermore, Lebesgue's differentiation theorem gives us that $\mathcal{L}^n(\Sigma_{1,\Omega} \cup \Sigma_{2,\Omega}) = 0$. □

The boundary setting

We now turn our attention to the model half-ball, where we provide analogous calculations to characterise singular points along the boundary. Given that the boundary itself has Lebesgue measure zero we do not obtain a more meaningful estimate on the size of this set. However, this characterisation is performed with generalisations of the dimension reduction arguments found in [DKM07,Bec11a] in mind. The structure of the proof itself is similar to the interior setting, with a few technical modifications.

Proposition 6.14. *Let* $u \in W^{1,p(\,\cdot\,)}_{\Gamma}\big(B^+, \mathbb{R}^N\big)$ *be a weak solution to*

$$-\operatorname{div} a(x,u,Du) = b(x,u,Du) \quad \textit{in} \;\; B^+,$$

under assumptions **(H1)–(H6)**, *where the inhomogeneity* b *satisfies either* **(B1)**, *or* **(B2)** *under the additional assumption that the solution is bounded with* $2L_1\|u\|_{L^\infty} < \nu$. *Then* $\operatorname{Reg}_{Du}(\Gamma)$ *is relatively open in* Γ *and for every* $x_0 \in \operatorname{Reg}_{Du}(\Gamma)$, Du *is in* $C^{1,\alpha}$ *in some relative neighbourhood of* x_0 *in* $\Gamma \cup B^+$.

Proof of Proposition 6.14: Taking $x_0 \in \Gamma_0 = \Gamma \setminus (\Sigma_{1,\Gamma} \cup \Sigma_{2,\Gamma})$ we fix

$$M := 1 + 2\limsup_{\rho \downarrow 0} \big|(D_n u)^+_{x_0,\rho}\big|,$$

and again infer the existence of a $0 < \hat\rho < \min\{\rho_0, (1-|x_0|)/4\}$ with $B^+_{4\hat\rho}(x_0) \subset B^+$, and $|(D_n u)^+_{x_0,\hat\rho}| < \frac{M}{2}$. We will consider points in three distinct cases, each with slightly differing estimates. As such, we require a further restriction on the parameters from Lemma (6.12), namely that

$$c(p_2)^2 2^n 3^n \bigg(\hat\rho + \mathcal{C}(x_0,\hat\rho)\bigg) < \bar\varepsilon \quad \text{and} \quad c(p_2)^2 2^{2n} 3^{n+2} c_d \sqrt{\bar\varepsilon} \le \frac{M}{4}, \tag{6.29}$$

where $c(p_2)$ is the constant appearing in Lemma 3.8. Of course the maps

$$x \mapsto \mathcal{C}(x,\rho), \quad \text{and} \quad x \mapsto (D_n u)^+_{x,\hat\rho},$$

are both continuous, so condition (6.19) continues to hold for each point in a neighbourhood in $\Gamma^+_{6R}(x_0) \cup B^+_{6R}(x_0)$, for some $0 < R \le \frac{\hat\rho}{12}$. By the same argument, we have that the bound

$$\big|(D_n u)^+_{x,\rho}\big| < \frac{M}{2} \tag{6.30}$$

will hold in the same neighbourhood, for all $\rho < 6R$. Without loss of generality we set $x_0 = 0$. As in the interior, we restrict θ small enough to fix $2\tilde\beta = 1 + \sigma$, where $\sigma = \min\{\frac{\alpha}{2}, \frac{\alpha}{p_2}\}$. This ensures that the conditions of Lemma 6.12 are satisfied, and so for any $x \in \Gamma^+_{6R} \cup B^+_{6R}$

$$\mathcal{C}(x,r) \le c_d \left[\left(\frac{r}{6R}\right)^{1+\sigma} \mathcal{C}(x,6R) + r^{2\sigma} \right]. \tag{6.31}$$

We now consider several cases, compare with [Bec08, Gro02b].

Case 1: $y \in \Gamma_{2R}, 0 < |y| \le \rho \le 4R$:

We simply use the inclusion $B^+_\rho(y) \subset B^+_{\rho+|y|}(0)$ and set $r = \rho + |y| \le 6R$ in (6.31). This tells us via Lemma 3.8 and Lemma 6.12 that

$$
\begin{aligned}
\mathcal{C}(y,\rho) &= \fint_{B_\rho^+(y)} \left|V\big(Du-(D_nu)_{y,\rho}^+\otimes e_n\big)\right|^2 dx \\
&\le c(p_2)\fint_{B_\rho^+(y)} \left|V\big(Du-(D_nu)_{0,\rho+|y|}^+\otimes e_n\big)\right|^2 dx \\
&\le c(p_2)\left(\frac{\rho+|y|}{\rho}\right)^n \fint_{B_{\rho+|y|}^+} \left|V\big(Du-(D_nu)_{0,\rho+|y|}^+\otimes e_n\big)\right|^2 dx \\
&\le c(p_2)2^n\mathcal{C}(0,\rho+|y|) \\
&\le c(p_2)2^n c_d\left[\left(\frac{\rho+|y|}{6R}\right)^{1+\sigma}\mathcal{C}(0,6R)+(\rho+|y|)^{2\sigma}\right] \\
&\le c(p_2)2^{n+2} c_d\left[\left(\frac{\rho}{6R}\right)^{1+\sigma}\mathcal{C}(0,6R)+\rho^{2\sigma}\right].
\end{aligned}
\tag{6.32}
$$

Case 2: $y\in\Gamma_{2R}, 0<\rho<|y|<2R$:

In this case we begin by checking

$$
\begin{aligned}
\mathcal{C}(y,2R) &= \fint_{B_{2R}^+(y)} \left|V\big(Du-(D_nu)_{y,2R}^+\otimes e_n\big)\right|^2 dx \\
&\le c(p_2)\fint_{B_{2R}^+(y)} \left|V\big(Du-(D_nu)_{0,6R}^+\otimes e_n\big)\right|^2 dx \\
&\le c(p_2)\left(\frac{2R}{6R}\right)^n \fint_{B_{6R}^+} \left|V\big(Du-(D_nu)_{0,6R}^+\otimes e_n\big)\right|^2 dx \\
&= c(p_2)3^n\mathcal{C}(0,6R),
\end{aligned}
\tag{6.33}
$$

and so we have that $\mathcal{C}(y,2R)+2R$ satisfies (6.29).

This allows us to easily infer from Lemma 6.12 via (6.31)

$$
\begin{aligned}
\mathcal{C}(y,\rho) &\le c_d\left[\left(\frac{\rho}{2R}\right)^{1+\sigma}\mathcal{C}(y,2R)+\rho^{2\sigma}\right] \\
&\le c(p_2)3^n c_d\left[\left(\frac{\rho}{2R}\right)^{1+\sigma}\mathcal{C}(0,6R)+\rho^{2\sigma}\right] \\
&\le c(p_2)3^{n+2} c_d\left[\left(\frac{\rho}{6R}\right)^{1+\sigma}\mathcal{C}(0,6R)+\rho^{2\sigma}\right].
\end{aligned}
\tag{6.34}
$$

Case 3: $y\in B_{2R}^+, B_\rho(y)\subset B_{2R}^+$:

Writing $y' = (y_1, ..., y_{n-1}, 0)$ for the projection of B^+ onto Γ, we have that

$$B_\rho(y) \subset B_{y_n}(y) \subset B^+_{2y_n}(y') \subset B^+_{2R}(y').$$

Now, if $|y'| \le 2y_n \le 4R$ then

$$\begin{aligned}\fint_{B_{y_n}(y)} \left|V\big(Du - (D_n u)_{y,y_n} \otimes e_n\big)\right|^2 dx &\le c(p_2) \fint_{B_{y_n}(y)} \left|V\big(Du - (D_n u)^+_{y',2y_n} \otimes e_n\big)\right|^2 dx \\ &\le c(p_2) 2^n \fint_{B^+_{2y_n}(y')} \left|V\big(Du - (D_n u)^+_{y',2y_n} \otimes e_n\big)\right|^2 dx,\end{aligned} \tag{6.35}$$

which is reduced to Case 1. Following the same argument, Lemma 6.12 combined with (6.35) gives

$$\fint_{B_{y_n}(y)} \left|V\big(Du - (D_n u)_{y,y_n} \otimes e_n\big)\right|^2 dx \le c(p_2)^2 2^{2n+2} c_d \left[\left(\frac{y_n}{6R}\right)^{1+\sigma} \mathcal{C}(0, 6R) + y_n^{2\sigma}\right].$$

On the other hand, if $2y_n < |y'| < 2R$, then (6.35) lets us apply Case 2, where $c(p_2)^2 2^n 3^n$ replaces $c(p_2)3^n$ in (6.33) and hence (6.34). So $\mathcal{C}(y, 2R) + 2R$ still satisfies (6.29), and we obtain

$$\fint_{B_{y_n}(y)} \left|V\big(Du - (D_n u)_{y,y_n} \otimes e_n\big)\right|^2 dx \le c(p_2)^2 2^n 3^{n+2} c_d \left[\left(\frac{y_n}{6R}\right)^{1+\sigma} \mathcal{C}(0, 6R) + y_n^{2\sigma}\right]. \tag{6.36}$$

Note that in either case we have (6.36). We now wish to apply the interior version of Lemma 6.12 on $B_{y_n}(y)$. However, this requires suitable control of the mean value of u, since this term appears in (6.17) but not (6.18). In particular, we can apply Poincaré's inequality then Lemma 3.4 (iv), keeping in mind that $y_n < \frac{1}{2}$, to compute

$$\begin{aligned}
\left|(u)_{y,y_n}\right| &\leq \fint_{B_{y_n}(y)} \left|u - (D_n u)^+_{0,2y_n} x_n\right| dx + \fint_{B_{y_n}(y)} \left|(D_n u)^+_{0,2y_n} x_n\right| dx \\
&\leq 2^n \fint_{B^+_{2y_n}} \left|u - (D_n u)^+_{0,2y_n} x_n\right| dx + \fint_{B_{y_n}(y)} \left|(D_n u)^+_{0,2y_n} x_n\right| dx \\
&\leq 2^{n+1} y_n \fint_{B^+_{2y_n}} \left|Du - (D_n u)^+_{0,2y_n} \otimes e_n\right| dx + \left|(D_n u)^+_{0,2y_n}\right| y_n \\
&\leq 2^{n-1} \left(\mathcal{C}(0, 2y_n) + \mathcal{C}^{\frac{1}{2}}(0, 2y_n)\right) + \frac{1}{4}\left|(D_n u)^+_{0,2y_n}\right| \\
&\leq 2^{n-1} \left(c(p_2)^2 2^n 3^{n+2} c_d \bar{\varepsilon} + \left(c(p_2)^2 2^n 3^{n+2} c_d \bar{\varepsilon}\right)^{\frac{1}{2}}\right) + \frac{1}{4}\left|(D_n u)_{0,2y_n}\right| \\
&\leq c(p_2)^2 2^{2n} 3^{n+2} c_d \sqrt{\bar{\varepsilon}} + \frac{M}{8} \\
&\leq \frac{M}{2},
\end{aligned}$$

provided (6.29) and (6.30) hold. So the conditions for the interior version of Lemma 6.12 are satisfied on $B_{y_n}(y)$, and we conclude via Lemma 6.12 and (6.36) that

$$\begin{aligned}
&\fint_{B_r(y)} \left|V\left(Du - (D_n u)_{y,r} \otimes e_n\right)\right|^2 dx \\
&\qquad \leq c_d \left[\left(\frac{r}{y_n}\right)^{1+\sigma} \fint_{B_{y_n}(y)} \left|V\left(Du - (D_n u)_{y,y_n} \otimes e_n\right)\right|^2 dx + r^{2\sigma}\right] \\
&\qquad \leq c(p_2)^2 2^n 3^{n+2} c_d^2 \left[\left(\frac{r}{6R}\right)^{1+\sigma} \mathcal{C}(0, 6R) + r^{2\sigma}\right].
\end{aligned}$$

Regardless of the case considered, we can combine this last estimate with (6.32) and (6.34), together with a suitable analogue of (6.26) in the superquadratic case, and (6.27) in the subquadratic case, to once again invoke Theorem 2.1. We now use (6.28) to obtain as before that Du has a continuous representative with $u \in C^{1,\sigma}(\overline{B^+_R}, \mathbb{R}^N)$.

Once we know that Du is Hölder continuous, its boundedness on $B^+_R(x_0)$ again implies the Lipschitz continuity of u. So Remark 6.5 and Remark 6.8 in Remark 6.11, lets us repeat the above argument to conclude $u \in C^{1,\alpha}(B^+_R(x_0), \mathbb{R}^N)$. We note as before that any dependence on p_2 is continuous, and the constants remain bounded when their dependence on p_2 is replaced globally by γ_1 and γ_2. $\square$

The transformation from the model half-ball setting to the boundary of Ω is now standard in view of Chapter 5, and we refer the reader to §6.3.6 of [Bec08] for detailed calculations. This shows Theorem 6.1. $\blacksquare$

Chapter 7

Systems with discontinuous coefficients

This chapter contains the proof of Theorem 2.3. We will prove interior partial Hölder continuity of solutions to systems of nonlinear elliptic PDE in divergence form, where the coefficients may be discontinuous. In particular, we are concerned with weak solutions to the problem

$$-\operatorname{div} a(x,u,Du) = b(x,u,Du) \quad \text{in} \;\; \Omega. \tag{7.1}$$

Here a weak solution is interpreted, in the usual sense, as any function $u \in W^{1,p(\,\cdot\,)}(\Omega,\mathbb{R}^N)$ satisfying

$$\int_\Omega a(x,u,Du)D\phi\,dx = \int_\Omega b(x,u,Du)\phi\,dx$$

for any fixed $\phi \in W_0^{1,p(\,\cdot\,)}(\Omega,\mathbb{R}^N)$.

We assume the familiar log-Hölder continuity condition on the exponent function $p : \Omega \to [\gamma_1,\gamma_2]$. That is, for some $L > 0$ there holds

$$\limsup_{\rho\downarrow 0} \omega_p(\rho)\log\left(\frac{1}{\rho}\right) \le L. \tag{7.2}$$

Furthermore we have the natural energy bound on the solution

$$\int_\Omega |Du|^{p(x)}\,dx \le E < \infty. \tag{7.3}$$

We assume $a : \Omega\times\mathbb{R}^N\times\mathrm{Hom}(\mathbb{R}^n\otimes\mathbb{R}^N)$ is a Borel measurable Carathédory vector field, whose partial map $z \mapsto a(\,\cdot\,,\,\cdot\,,z)$ is differentiable. Furthermore, we assume ellipticity and nonstandard growth conditions are satisfied, for some $1 < \gamma_1 \le p(x) \le \gamma_2 < \infty$ and $0 < \nu \le 1 \le L < \infty$. That is,

(V1) $\nu(1+|z|)^{p(x)-2}|\zeta|^2 \le D_z a(x,\xi,z)\zeta\cdot\zeta$,

(V2) $|D_z a(x,\xi,z)| \le L(1+|z|)^{p(x)-2}$,

(V3) $|a(x,\xi,z)| \le L(1+|z|)^{p(x)-1}$, for all $(x, \xi, z) \in \Omega \times \mathbb{R}^N \times \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$ and $\zeta \in \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$.

We further assume the vector field a is continuous in its second variable u with bounded, concave, non-decreasing modulus of continuity ω, and its partial derivatives $D_z a$ are continuous with modulus of continuity μ. That is, $\omega_\xi, \mu : [0, \infty) \to [0, 1]$ satisfy $\lim_{r\downarrow 0} \omega_\xi(r) = 0$ and $\lim_{r\downarrow 0} \mu(r) = 0$, and

(V4) $\big|a(x, \xi, z) - a(x, \hat{\xi}, z)\big| \leq L\omega_\xi\big(|\xi - \hat{\xi}|\big)(1 + |z|)^{p(x)-1}$,

(V5) $\big|D_z a(x, \xi, z) - D_z a(x, \xi, \bar{z})\big| \leq \begin{cases} L\mu\left(\frac{|z-\bar{z}|}{1+|z|+|\bar{z}|}\right)\big(1 + |z| + |\bar{z}|\big)^{p(x)-2} & 2 \leq p(x) \\ L\mu\left(\frac{|z-\bar{z}|}{1+|z|+|\bar{z}|}\right)\left(\frac{1+|z|+|\bar{z}|}{(1+|z|)(1+|\bar{z}|)}\right)^{2-p(x)} & 1 < p(x) < 2. \end{cases}$

We make no assumptions regarding the continuity of the vector field a in its first variable, requiring only the VMO-type condition on the map $x \mapsto \frac{a(x,\xi,z)}{(1+|z|^2)^{p-1}}$. That is for $x_0 \in \Omega$, $r \in (0, \rho_0]$, $\xi \in \mathbb{R}^N$ and $z \in \mathbb{R}^{nN}$ there holds

(V6) $\big|a(x, \xi, z) - (a(x, \xi, z))_{x_0,r}\big| \leq \mathbf{v}_{x_0}(x, r)\Big[(1+|z|)^{p(x)-1} + (1+|z|)^{p(x_0)-1}\Big]\Big[1 + \log(1 + |z|)\Big]$

uniformly in ξ and z. Here, $\mathbf{v}_{x_0} : \mathbb{R}^n \times [0, \rho_0] \to [0, 2L]$ is a bounded function satisfying

(VMO) $\lim_{\rho \searrow 0} \mathbf{V}(\rho) = 0, \qquad$ where $\qquad \mathbf{V}(\rho) := \sup_{x_0 \in \Omega, r \in (0,\rho]} \fint_{B_r(x_0)\cap\Omega} \mathbf{v}_{x_0}(x, r)\, dx.$

Finally, we will assume the inhomogeneous term b has *controllable growth*, i.e.

(I) $b(x, \xi, z) \leq L(1 + |z|^2)^{p(x)-1}$.

Statement of main result

We are now in a position to state our main theorem.

Theorem 7.1. *Let $u \in W^{1,p(\,\cdot\,)}\big(\Omega, \mathbb{R}^N\big)$ be a weak solution to* (7.1) *under assumptions* **(V1)–(V6)**, *where the inhomogeneity b satisfies* **(I)**. *Then there exist $\kappa, \sigma > 0$ such that the following hold:*

(i) Ω_u and Ω_u^α are relatively open in Ω, for each $\alpha \in (0, 1)$,

(ii) $u \in C^{0,\alpha}\big(\Omega_u, \mathbb{R}^N\big)$ for every $\alpha \in (0, 1)$, and
for every $\alpha \in (0, 1)$ $u \in C^{0,\alpha}\big(\Omega_u^\alpha, \mathbb{R}^N\big)$,

(iii) $\Omega_u \subset \Big(\Sigma^0_{1,\Omega} \cup \Sigma^0_{2,\Omega} \cup \Sigma_{3,\Omega}\Big)$, and $\Omega_u^\alpha \subset \Big(\Sigma^\kappa_{1,\Omega} \cup \Sigma^\sigma_{2,\Omega} \cup \Sigma_{3,\Omega}\Big)$, with these sets defined in (2.10), (2.11) *and* (2.12).

In particular, we have $\mathcal{L}^n(\Omega \setminus \Omega_u) = 0$.

A technical lemma

Since we are ultimately using a localised freezing technique, it is convenient to consider estimates in terms of a fixed exponent. To treat both the variable exponent and VMO conditions simultaneously, we require the following lemma. The parabolic version of this estimate appears as Lemma 3.2 in [DH12]. A similar elliptic counterpart can be retrieved from the proof of Lemma 3.4 in [BDHS12]. We adapt their proof to our choice of exponent.

Lemma 7.2. *Let $M > 0$ and assume that $u \in W^{1,p(\,\cdot\,)}(\Omega, \mathbb{R}^N)$ is a weak solution to* (7.1) *under structure conditions* **(V1)**–**(V6)**, *with the inhomogeneity satisfying* **(I)**, (7.2) *and* (7.3). *Fix $D\ell \in \mathbb{R}^{nN}$ to satisfy $\Phi(x_0, D\ell, \theta\rho) \leq \frac{1}{36}$ and $|D\ell| \leq M$, and take smooth some $\eta \in C_c^\infty(B_\rho(x_0))$. Then*

$$\fint_{B_\rho(x_0)} \eta^{\hat{p}} \Big| \big(1 + |D\ell|^2 + |Du|^2\big)^{\frac{p(x)-2}{2}} - \big(1 + |D\ell|^2 + |Du|^2\big)^{\frac{p_2-2}{2}} \Big| \big|Du - D\ell\big|^2 \, dx$$
$$\leq \frac{1}{2} \int_{B_\rho(x_0)} \eta^{\hat{p}} \big(1 + |D\ell|^2 + |Du|^2\big)^{\frac{p_2-2}{2}} \big|Du - D\ell\big|^2 \, dx + c\big(1 + |D\ell|\big)^{p_2} \omega_p(\rho).$$

Here, the constant depends on $n, N, L/\nu, \gamma_1, \gamma_2, M, \omega_p, L_1, L_2, E$ and ω_p.

Proof of Lemma 7.2: Noting that $p(x) \leq p_2$ in the domain, with $p_2 - p(x) \leq \omega(\rho)$, we have for $y \geq 0$ the pointwise estimate

$$\Big| \big(1 + y\big)^{\frac{p(x)-2}{2}} - \big(1 + y\big)^{\frac{p_2-2}{2}} \Big| = \Big| \frac{(p(x) - p_2)}{2} \int_0^1 \big(1 + y\big)^{\frac{sp(x)+(1-s)p_2-2}{2}} \log(1 + y) \, ds \Big|$$
$$\leq \frac{1}{2} \omega_p(\rho)(1 + y)^{\frac{p_2-2}{2}} \log(1 + y).$$

Consequently, we have via Young's inequality with exponents $(2, 2)$

$$\fint_{B_\rho(x_0)} \eta^{\hat{p}} \Big| \big(1 + |D\ell|^2 + |Du|^2\big)^{\frac{p(x)-2}{2}} - \big(1 + |D\ell|^2 + |Du|^2\big)^{\frac{p_2-2}{2}} \Big| \big|Du - D\ell\big|^2 \, dx$$
$$\leq c\omega_p(\rho) \fint_{B_\rho(x_0)} \eta^{\hat{p}} \big(1 + |D\ell|^2 + |Du|^2\big)^{\frac{p_2-2}{4} + \frac{p_2}{2}} \log\big(1 + |D\ell|^2 + |Du|^2\big) \big|Du - D\ell\big| \, dx$$
$$\leq \frac{1}{2} \fint_{B_\rho(x_0)} \eta^{\hat{p}} \big(1 + |D\ell|^2 + |Du|^2\big)^{\frac{p_2-2}{2}} |Du - D\ell|^2 \, dx$$
$$+ c\omega_p(\rho) \fint_{B_\rho(x_0)} \eta^{\hat{p}} (1 + |D\ell| + |Du|)^{p_2} \log\big(1 + |D\ell| + |Du|\big) \, dx$$

In handling the second term, we easily have when $|Du| \leq |D\ell|$, we easily have via the inequality $\log(1+|z|) \leq C(\delta)|z|^{\delta}$

$$\begin{aligned}
\omega_p(\rho) \fint_{B_\rho(x_0)} & \eta^{\hat{p}}(1+|D\ell|+|Du|)^{p_2} \log\big(1+|D\ell|+|Du|\big)\chi_{(|Du|\leq|D\ell|)}\,dx \\
&\leq c\omega_p(\rho)(1+|D\ell|)^{p_2}\log\big(1+|D\ell|\big) \\
&\leq c(\delta)\omega_p(\rho)(1+|D\ell|)^{p_2(1+\hat{\delta})} \\
&\leq c(\delta,M)\omega_p(\rho)(1+|D\ell|)^{p_2}.
\end{aligned}$$

On the other hand, when $|D\ell| < |Du|$ we again apply the inequality $\log(1+|z|) \leq C(\delta)|z|^{\delta}$, then Corollary 4.2 and Corollary 4.3 (i) to find

$$\begin{aligned}
\omega_p(\rho) \fint_{B_\rho(x_0)} & \eta^{\hat{p}}(1+|D\ell|+|Du|)^{p_2} \log\big(1+|D\ell|+|Du|\big)\chi_{(|D\ell|<|Du|)}\,dx \\
&\leq c\omega_p(\rho)\fint_{B_\rho(x_0)} \eta^{\hat{p}}(1+|Du|)^{p_2}\log\big(1+|Du|\big)\,dx \\
&\leq c(\delta)\omega_p(\rho)\fint_{B_\rho(x_0)} (1+|Du|)^{p_2(1+\frac{\hat{\delta}}{4})}\,dx \\
&\leq c(\delta)\omega_p(\rho)\bigg(\fint_{B_{2\rho}(x_0)} (1+|Du|)^{p_2}\,dx\bigg)^{1+\hat{\delta}} \\
&\leq c(\delta)\omega_p(\rho)(1+|D\ell|)^{p_2(1+\hat{\delta})} \\
&\leq c(\delta,M)\omega_p(\rho)(1+|D\ell|)^{p_2}.
\end{aligned}$$

Noting that the constant has inherited all of the dependences from Corollary 4.2, we combine these cases to conclude the result. □

A Caccioppoli inequality

As in the previous chapter, our first step is to establish a suitable Caccioppoli, or reverse Poincaré inequality. This allows us to control an integral in the gradient in terms of one in the solution, at a local scale.

Lemma 7.3 (Caccioppoli Inequality). *Let $u \in W^{1,p(\,\cdot\,)}(\Omega,\mathbb{R}^N)$ be a weak solution to* (7.1) *with* (7.2) *and* (7.3)*, under structure conditions* **(V1)**–**(V6)***, with the inhomogeneity satisfying* **(I)***. Fix an affine function $\ell : \mathbb{R}^n \to \mathbb{R}^N$ satisfying $\Phi(x_0, D\ell, \rho) \leq \frac{1}{36}$ and $|D\ell| \leq M$ for some $M > 0$. Then there exist constants $\rho_0 = \rho_0(n,N,L/\nu,\gamma_1,\gamma_2,L_1,L_2,\omega_p) << 1$ and $c = c_c(n,N,L/\nu,\gamma_1,\gamma_2,L_1,L_2,E,\omega_p,M)$ such that for every $\rho < \rho_0$ and any ball*

$B_\rho(x_0) \subset\subset \Omega$ *with* $p_2 = \sup_{B_\rho(x_0)} p(\,\cdot\,)$ *and* $V = V_{p_2}$*, the following estimate holds:*

$$\fint_{B_{\frac{\rho}{2}}(x_0)} \left| V\left(\frac{Du - D\ell}{1 + |D\ell|}\right)\right|^2 dx \le c_c \left(\fint_{B_\rho(x_0)} \left| V\left(\frac{u - \ell(x)}{\rho(1 + |D\ell|)}\right)\right|^2 dx \right.$$
$$\left. + \omega_\xi^2\left(\fint_{B_\rho(x_0)} |u - \ell(x_0)|\, dx\right) + \mathbf{V}(\rho) + \omega_p(\rho) + \rho^2 \right).$$

Proof of Lemma 7.3: Taking a standard cutoff function $\eta \in C_0^\infty(B_\rho(x_0))$ that satisfies $0 \le \eta \le 1$, $\eta = 1$ on $B_{\frac{\rho}{2}}(x_0)$, $\eta = 0$ outside $B_{\frac{3\rho}{4}}(x_0)$ and $|D\eta| \le \frac{C}{\rho}$. We write $\phi := \eta^{\hat{p}} w$ for $\hat{p} = \max\{2, p_2\}$ and $w := u - \ell(x)$. Then $\phi \in W_0^{1,p(\,\cdot\,)}(B_\rho(x_0), \mathbb{R}^N)$, with

$$D\phi = \hat{p}\eta^{\hat{p}-1}\big(u - \ell(x)\big) \otimes D\eta + \eta^{\hat{p}}(Du - D\ell). \tag{7.4}$$

Since u solves (7.1) there holds

$$\fint_{B_\rho(x_0)} a(x, u, Du) \cdot D\phi\, dx \;=\; \fint_{B_\rho(x_0)} b(x, u, Du) \cdot \phi\, dx,$$

and trivially

$$\fint_{B_\rho(x_0)} \big(a(\,\cdot\,, \ell(x_0), D\ell)\big)_{\rho, x_0} \cdot D\phi\, dx \;=\; 0.$$

We proceed to calculate

$$\begin{aligned}
&\fint_{B_\rho(x_0)} \eta^{\hat{p}}\big[a(x, u, Du) - a(x, u, D\ell)\big] \cdot (Du - D\ell)\, dx \\
&\qquad = \fint_{B_\rho(x_0)} \Big[\big(a(\,\cdot\,, \ell(x_0), D\ell)\big)_{\rho, x_0} - a(x, \ell(x_0), D\ell)\Big] \cdot D\phi\, dx \\
&\qquad\qquad + \fint_{B_\rho(x_0)} \big[a(x, \ell(x_0), D\ell) - a(x, u, D\ell)\big] \cdot D\phi\, dx \\
&\qquad\qquad + \hat{p}\fint_{B_\rho(x_0)} \eta^{\hat{p}-1}\big[a(x, u, Du) - a(x, u, D\ell)\big] \cdot w \otimes D\eta\, dx \\
&\qquad\qquad + \fint_{B_\rho(x_0)} b(x, u, Du) \cdot \eta^{\hat{p}} w,
\end{aligned}$$

with the obvious labelling

$$\mathrm{I} = \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}.$$

We now consider each term independently. We can use elementary integration with **(V1)** and Lemma 3.6, then Lemma 7.2 with Lemma 3.5 to estimate

$$\begin{aligned}
\mathrm{I} &= \fint_{B_\rho(x_0)} \eta^{\hat{p}} \big[a(x,u,Du) - a(x,u,D\ell)\big] \cdot (Du - D\ell)\, dx \\
&= \fint_{B_\rho(x_0)} \eta^{\hat{p}} \int_0^1 \Big[D_z a\big(x,u,D\ell + t(Du - D\ell)\big)(Du - D\ell)\Big] \cdot (Du - D\ell)\, dt\, dx \\
&\geq c \fint_{B_\rho(x_0)} \eta^{\hat{p}} \int_0^1 \big(1 + |D\ell + t(Du - D\ell)|\big)^{p(x)-2} |Du - D\ell|^2\, dt\, dx \\
&\geq c \fint_{B_\rho(x_0)} \eta^{\hat{p}} \big(1 + |D\ell|^2 + |Du|^2\big)^{p(x)-2} |Du - D\ell|^2\, dx \\
&\geq c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}} \left| V\left(\frac{Du - D\ell}{1 + |D\ell|}\right)\right|^2 dx - (1+|D\ell|)^{p_2} \omega_p^2(\rho).
\end{aligned}$$

Here the first constant already has the dependences of the constant from Lemma 7.2, and will gain no additional dependences. Note that the superquadratic and subquadratic cases differ only in the value of the constants.

Considering the second term, we calculate via (7.4) that

$$\begin{aligned}
\mathrm{II} &= \fint_{B_\rho(x_0)} \Big[\big(a(\,\cdot\,, \ell(x_0), D\ell)\big)_{x_0,\rho} - a(x, \ell(x_0), D\ell)\Big] \cdot D\phi\, dx \\
&= \fint_{B_\rho(x_0)} \hat{p}\eta^{\hat{p}-1} \Big[\big(a(\,\cdot\,, \ell(x_0), D\ell)\big)_{x_0,\rho} - a(x, \ell(x_0), D\ell)\Big] \cdot w \otimes D\eta\, dx \\
&\qquad + \fint_{B_\rho(x_0)} \eta^{\hat{p}} \Big[\big(a(\,\cdot\,, \ell(x_0), D\ell)\big)_{x_0,\rho} - a(x, \ell(x_0), D\ell)\Big] \cdot (Du - D\ell)\, dx \\
&= \mathrm{II}_a + \mathrm{II}_b,
\end{aligned}$$

again with the obvious labelling.

Define the set $S_- := \big\{x \in B_\rho(x_0) : |Du - D\ell| < 1 + |D\ell|\big\}$ and $S_+ := B_\rho(x_0) \setminus S_-$, and denote to be the characteristic function of the set S by $\chi_{(S)}$. Now we can use **(V6)** along with Young's inequality (with exponent pairs $(2,2)$ and $(p_2, \frac{p_2}{p_2-1})$), keeping in mind $0 \leq \mathbf{v}_{x_0} \leq 2L$, Hölder's inequality (with exponents $(2,2)$), Lemma 3.4 (iv) and the bound

$|D\ell| \le M$ to compute

$$\begin{aligned}
\mathrm{II}_b &\le \fint_{B_\rho(x_0)} \eta^{\hat p}\big|\big(a(\,\cdot\,,\ell(x_0),D\ell)\big)_{x_0,\rho} - a(x,\ell(x_0),D\ell)\big||Du-D\ell|\,dx \\
&\le c\fint_{B_\rho(x_0)} \eta^{\hat p}\mathbf{v}_{x_0}(x,\rho)(1+|D\ell|)^{p_2-1}\big[1+\log(1+|D\ell|)\big]|Du-D\ell|\,dx \\
&\le c(1+|D\ell|)^{p_2}\big[1+\log(1+|D\ell|)\big]\fint_{B_\rho(x_0)} \eta^{\hat p}\mathbf{v}_{x_0}(x,\rho)\left|\frac{Du-D\ell}{(1+|D\ell|)}\right|dx \\
&\le c(\varepsilon,M)(1+|D\ell|)^{p_2}\fint_{B_\rho(x_0)} \mathbf{v}_{x_0}^{\frac{p_2}{p_2-1}}(x,\rho)\chi_{(S_+)} + \mathbf{v}_{x_0}^2(x,\rho)\chi_{(S_-)}\,dx \\
&\qquad + \varepsilon(1+|D\ell|)^{p_2}\fint_{B_\rho(x_0)} \eta^{\hat p}\left|\frac{Du-D\ell}{(1+|D\ell|)}\right|^2\chi_{(S_-)} + \eta^{\hat p}\left|\frac{Du-D\ell}{(1+|D\ell|)}\right|^{p_2}\chi_{(S_+)}\,dx \\
&\le c(\varepsilon,M)(1+|D\ell|)^{p_2}\mathbf{V}(\rho) + \varepsilon(1+|D\ell|)^{p_2}\fint_{B_\rho(x_0)} \eta^{\hat p}\left|V\left(\frac{Du-D\ell}{1+|D\ell|}\right)\right|dx.
\end{aligned}$$

Here we have left ε to be chosen later, and have used the fact that for $q>1$ and $\mathbf{v}\le 2L$

$$\mathbf{v}_{x_0}^q(x,\rho) \le (2L)^{q-1}\mathbf{v}_{x_0}(x,\rho).$$

We can estimate II_a in the same way, instead considering the sets

$$T_- := \big\{x\in B_\rho(x_0) : |w| < (1+|D\ell|)\rho\big\} \qquad \text{and} \qquad T_+ := B_\rho(x_0)\setminus T_-,$$

and calculate via the same process

$$\begin{aligned}
\mathrm{II}_a &\le \fint_{B_\rho(x_0)} \eta^{\hat p}\big|\big(a(\,\cdot\,,\ell(x_0),D\ell)\big)_{x_0,\rho} - a(x,\ell(x_0),D\ell)\big|\left|\frac{w}{\rho}\right|dx \\
&\le c\fint_{B_\rho(x_0)} \eta^{\hat p}\mathbf{v}_{x_0}(x,\rho)(1+|D\ell|)^{p_2-1}\big[1+\log(1+|D\ell|)\big]\left|\frac{w}{\rho}\right|dx \\
&\le c(1+|D\ell|)^{p_2}\big[1+\log(1+|D\ell|)\big]\fint_{B_\rho(x_0)} \eta^{\hat p}\mathbf{v}_{x_0}(x,\rho)\left|\frac{w}{(1+|D\ell|)\rho}\right|dx \\
&\le c(1+|D\ell|)^{p_2}\fint_{B_\rho(x_0)} \mathbf{v}_{x_0}^{\frac{p_2}{p_2-1}}(x,\rho)\chi_{(T_+)} + \mathbf{v}_{x_0}^2(x,\rho)\chi_{(T_-)}\,dx \\
&\qquad + c(1+|D\ell|)^{p_2}\fint_{B_\rho(x_0)} \eta^{\hat p}\left|\frac{w}{(1+|D\ell|)\rho}\right|^2\chi_{(T_-)} + \eta^{\hat p}\left|\frac{w}{(1+|D\ell|)\rho}\right|^{p_2}\chi_{(T_+)}\,dx \\
&\le c(1+|D\ell|)^{p_2}\left(\mathbf{V}(\rho) + \fint_{B_\rho(x_0)}\left|V\left(\frac{w}{(1+|D\ell|)\rho}\right)\right|dx\right).
\end{aligned}$$

To estimate III we notice that owing to **(V4)** and (7.4) there holds

$$\begin{aligned}
\mathrm{III} &= \fint_{B_\rho(x_0)} \big[a(x,\ell(x_0),D\ell) - a(x,u,D\ell)\big]\cdot D\phi\,dx \\
&\le L \fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big)(1+|D\ell|)^{p(x)-1}|D\phi|\,dx \\
&\le c(1+|D\ell|)^{p_2-1}\fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big)\left|\frac{w}{\rho}\right| dx \\
&\qquad + c(1+|D\ell|)^{p_2-1}\fint_{B_\rho(x_0)} \eta^{\hat p}\omega_\xi\big(|u-\ell(x_0)|\big)\big|Du-D\ell\big|\,dx \\
&\le c(1+|D\ell|)^{p_2}\fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big)\left|\frac{w}{(1+|D\ell|)\rho}\right| dx \\
&\qquad + c(1+|D\ell|)^{p_2}\fint_{B_\rho(x_0)} \eta^{\hat p}\omega_\xi\big(|u-\ell(x_0)|\big)\left|\frac{Du-D\ell}{1+|D\ell|}\right| dx \\
&= \mathrm{III}_a + \mathrm{III}_b.
\end{aligned}$$

with the obvious notation. Now when $2 \le p_2 < \infty$ we can use Young's inequality with exponents $(2,2)$ and Lemma 3.4 (iv) to deduce

$$\begin{aligned}
\mathrm{III}_a + \mathrm{III}_b &\le c(\varepsilon)(1+|D\ell|)^{p_2}\fint_{B_\rho(x_0)} \omega_\xi^2\big(|u-\ell(x_0)|\big) + \left|\frac{w}{(1+|D\ell|)\rho}\right|^2 dx \\
&\qquad + \varepsilon(1+|D\ell|)^{p_2}\fint_{B_\rho(x_0)} \eta^{2\hat p}\left|\frac{Du-D\ell}{1+|D\ell|}\right|^2 dx \\
&\le c(\varepsilon)(1+|D\ell|)^{p_2}\fint_{B_\rho(x_0)} \omega_\xi^2\big(|u-\ell(x_0)|\big) + \left|V\left(\frac{w}{(1+|D\ell|)\rho}\right)\right|^2 dx \\
&\qquad + \varepsilon(1+|D\ell|)^{p_2}\fint_{B_\rho(x_0)} \eta^{\hat p}\left|V\left(\frac{Du-D\ell}{1+|D\ell|}\right)\right|^2 dx.
\end{aligned}$$

To treat the subquadratic case, we recall the definition of S_- and S_+ from estimate II. We will consider III_b, with the calculations for III_a being completely analogous, again replacing S_- with T_-, and S_+ with T_+ as we have already done in II. For $\big|\frac{Du-D\ell}{(1+|D\ell|)}\big| < 1$ we find using Young's inequality (with both exponent pairs $(2,2)$ and $(p_2, \frac{p_2}{p_2-1})$), and

Lemma 3.4 (iv), keeping in mind that $\omega_\xi \le 1$ and $\frac{p_2}{p_2-1} > 2$:

$$\begin{aligned}
\mathrm{III}_b &= c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^2 \omega_\xi\big(|u-\ell(x_0)|\big)\left|\frac{Du-D\ell}{1+|D\ell|}\right| dx \\
&\le c(\varepsilon)(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \omega_\xi^2\big(|u-\ell(x_0)|\big)\chi_{(S_-)} + \omega_\xi^{\frac{p_2}{p_2-1}}\big(|u-\ell(x_0)|\big)\chi_{(S_+)}\, dx \\
&\qquad + (1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \varepsilon\eta^4 \left|\frac{Du-D\ell}{1+|D\ell|}\right|^2 \chi_{(S_-)} + \varepsilon\eta^{2p_2}\left|\frac{Du-D\ell}{1+|D\ell|}\right|^{p_2} \chi_{(S_+)}\, dx \\
&\le c(\varepsilon)(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big)\, dx \\
&\qquad + \varepsilon(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}} \left|V\left(\frac{Du-D\ell}{1+|D\ell|}\right)\right|^2 dx.
\end{aligned}$$

Analogously, for III_a we obtain

$$\mathrm{III}_a \le c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big) + \left|V\left(\frac{w}{1+|D\ell|}\right)\right|^2 dx.$$

Combining these estimates and applying Jensen's inequality yields

$$\begin{aligned}
\mathrm{III} &\le c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big) + \left|V\left(\frac{w}{(1+|D\ell|)\rho}\right)\right|^2 dx \\
&\qquad + \varepsilon(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}} \left|V\left(\frac{Du-D\ell}{1+|D\ell|}\right)\right|^2 dx \\
&\le c(1+|D\ell|)^{p_2} \omega_\xi\left(\fint_{B_\rho(x_0)} |u-\ell(x_0)|\, dx\right) \\
&\qquad + c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \left|V\left(\frac{w}{(1+|D\ell|)\rho}\right)\right|^2 dx \\
&\qquad + \varepsilon(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}} \left|V\left(\frac{Du-D\ell}{1+|D\ell|}\right)\right|^2 dx.
\end{aligned}$$

Estimating the next term in the case where $p_2 \geq 2$, we use **(V2)** to compute

$$
\begin{aligned}
\mathrm{IV} &= \hat{p} \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \big[a(x,u,Du) - a(x,u,D\ell)\big] \cdot w \otimes D\eta \, dx \\
&\leq c \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \int_0^1 \Big| D_z a\big(x,u,D\ell + t(Du - D\ell)\big)\Big| |Du - D\ell||w||D\eta| \, dt \, dx \\
&\leq c \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \left(1 + |D\ell + t(Du - D\ell)|\right)^{p_2-2} |Du - D\ell| \left|\frac{w}{\rho}\right| dx \\
&\leq c \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \left(1 + |D\ell| + |Du - D\ell|\right)^{p_2-2} |Du - D\ell| \left|\frac{w}{\rho}\right| dx
\end{aligned}
$$

Now we apply Young's inequality (with exponent pairs $(2,2)$ and $(p_2, \frac{p_2}{p_2-1})$), and finally Lemma 3.4 (iv) to calculate

$$
\begin{aligned}
\mathrm{IV} &\leq c \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \left(1 + |D\ell| + |Du - D\ell|\right)^{p_2-2} |Du - D\ell| \left|\frac{w}{\rho}\right| dx \\
&\leq c \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} (1 + |D\ell|)^{p_2-2} |Du - D\ell| \left|\frac{w}{\rho}\right| + \eta^{\hat{p}-1} |Du - D\ell|^{p_2-1} \left|\frac{w}{\rho}\right| dx \\
&\leq c(1 + |D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \left|\frac{Du - D\ell}{1 + |D\ell|}\right| \left|\frac{w}{\rho(1 + |D\ell|)}\right| dx \\
&\qquad + c(1 + |D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \left|\frac{Du - D\ell}{1 + |D\ell|}\right|^{p_2-1} \left|\frac{w}{\rho(1 + |D\ell|)}\right| dx \\
&\leq (1 + |D\ell|)^{p_2} \fint_{B_\rho(x_0)} \frac{\varepsilon}{2} \eta^{2(\hat{p}-1)} \left|\frac{Du - D\ell}{1 + |D\ell|}\right|^2 + C(\varepsilon) \left|\frac{w}{(1 + |D\ell|)\rho}\right|^2 dx \\
&\qquad + (1 + |D\ell|)^{p_2} \fint_{B_\rho(x_0)} \frac{\varepsilon}{2} \eta^{\hat{p}} \left|\frac{Du - D\ell}{1 + |D\ell|}\right|^{p_2} + C(\varepsilon) \left|\frac{w}{(1 + |D\ell|)\rho}\right|^{p_2} dx \\
&\leq \varepsilon (1 + |D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}} \left| V\left(\frac{Du - D\ell}{1 + |D\ell|}\right)\right|^2 dx \\
&\qquad + c(\varepsilon)(1 + |D\ell|)^{p_2} \fint_{B_\rho(x_0)} \left| V\left(\frac{w}{(1 + |D\ell|)\rho}\right)\right|^2 dx. \qquad (7.5)
\end{aligned}
$$

The setting with $1 < p_2 < 2$ is more delicate. We begin by applying **(V2)** and

Lemma 3.6 to compute

$$
\begin{aligned}
\mathrm{IV} &= \hat{p} \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \big[a(x,u,Du) - a(x,u,D\ell)\big] \cdot w \otimes D\eta \, dx \\
&\le c \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \int_0^1 \big|D_z a(x,u,D\ell + t(Du - D\ell)\big| |Du - D\ell||w||D\phi| \, dt \, dx \\
&\le c \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} (1 + |D\ell + t(Du - D\ell)|)^{p_2-2} |Du - D\ell| \left|\frac{w}{\rho}\right| dx \\
&\le c \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} (1 + |D\ell| + |Du - D\ell|)^{p_2-2} |Du - D\ell| \left|\frac{w}{\rho}\right| dx.
\end{aligned}
$$

Recalling the sets defined as $T_- = \{x \in B_\rho(x_0) : |w| < (1+|D\ell|)\rho\}$ and $S_- = \{x \in B_\rho(x_0) : |Du - D\ell| < (1 + |D\ell|)\}$, with $T_+ = B_\rho(x_0) \setminus T_-$ and $S_+ = B_\rho(x_0) \setminus S_-$, we now decompose the domain of integration into four parts

$$
\begin{aligned}
\mathrm{IV} \le\ & c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \left|\frac{Du - D\ell}{1+|D\ell|}\right|^{p_2-1} \left|\frac{w}{\rho(1+|D\ell|)}\right| \chi_{(T_+\cap S_+)} \, dx \\
&+ c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \left|\frac{Du - D\ell}{1+|D\ell|}\right| \left|\frac{w}{\rho(1+|D\ell|)}\right| \chi_{(T_-\cap S_-)} \, dx \\
&+ c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \left|\frac{w}{\rho(1+|D\ell|)}\right|^{p_2} \chi_{(T_+\cap S_-)} \, dx \\
&+ c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \left|\frac{Du - D\ell}{1+|D\ell|}\right|^{p_2-1} \left|\frac{w}{\rho(1+|D\ell|)}\right| \chi_{(T_-\cap S_+)} \, dx \\
=\ & \mathrm{IV}_a + \mathrm{IV}_b + \mathrm{IV}_c + \mathrm{IV}_d
\end{aligned} \tag{7.6}
$$

We first use Young's inequality and Lemma 3.4 (iv) to show

$$
\begin{aligned}
\mathrm{IV}_a &= c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \left|\frac{Du - D\ell}{1+|D\ell|}\right|^{p_2-1} \left|\frac{w}{\rho(1+|D\ell|)}\right| \chi_{(T_+\cap S_+)} \, dx \\
&\le (1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \varepsilon \eta^{\frac{p_2}{p_2-1}} \left|\frac{Du - D\ell}{1+|D\ell|}\right|^{p_2} \chi_{(T_+\cap S_+)} + c(\varepsilon) \left|\frac{w}{\rho(1+|D\ell|)}\right|^{p_2} \chi_{(T_+\cap S_+)} \, dx \\
&\le \varepsilon (1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}} \left|V\left(\frac{Du - D\ell}{1+|D\ell|}\right)\right|^2 \chi_{(T_+\cap S_+)} \, dx \\
&\qquad + c(\varepsilon)(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \left|V\left(\frac{w}{(1+|D\ell|)\rho}\right)\right|^2 \chi_{(T_+\cap S_+)} \, dx,
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\mathrm{IV}_b &= c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \left|\frac{Du-D\ell}{1+|D\ell|}\right| \left|\frac{w}{\rho(1+|D\ell|)}\right| \chi_{(T_-\cap S_-)}\,dx \\
&\le \varepsilon(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}} \left|\frac{Du-D\ell}{1+|D\ell|}\right|^2 \chi_{(T_-\cap S_-)}\,dx \\
&\qquad + c(\varepsilon)(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \left|\frac{w}{\rho(1+|D\ell|)}\right|^2 \chi_{(T_-\cap S_-)}\,dx \\
&\le \varepsilon(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}} \left|V\left(\frac{Du-D\ell}{1+|D\ell|}\right)\right|^2 \chi_{(T_-\cap S_-)}\,dx \\
&\qquad + c(\varepsilon)(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \left|V\left(\frac{w}{(1+|D\ell|)\rho}\right)\right|^2 \chi_{(T_-\cap S_-)}\,dx.
\end{aligned}
$$

For the third term we only need Lemma 3.4 (iv), since

$$
\begin{aligned}
\mathrm{IV}_c &= c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \left|\frac{w}{\rho(1+|D\ell|)}\right|^{p_2} \chi_{(T_+\cap S_-)}\,dx \\
&\le c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \left|V\left(\frac{w}{(1+|D\ell|)\rho}\right)\right|^2 \chi_{(T_+\cap S_-)}\,dx.
\end{aligned}
$$

Finally we use the fact that $p<2$ implies $2(p-1)<p$ and equivalently $\frac{p}{p-1}>2$, together with Young's inequality and Lemma 3.4 (iv) to find

$$
\begin{aligned}
\mathrm{IV}_d &= c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}-1} \left|\frac{Du-D\ell}{1+|D\ell|}\right|^{p_2-1} \left|\frac{w}{\rho(1+|D\ell|)}\right| \chi_{(T_-\cap S_+)}\,dx \\
&\le \varepsilon(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}} \left|\frac{Du-D\ell}{1+|D\ell|}\right|^{2(p_2-1)} \chi_{(T_-\cap S_+)}\,dx \\
&\qquad + c(\varepsilon)(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \left|\frac{w}{\rho(1+|D\ell|)}\right|^2 \chi_{(T_-\cap S_+)}\,dx \\
&\le \varepsilon(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat{p}} \left|V\left(\frac{Du-D\ell}{1+|D\ell|}\right)\right|^2 \chi_{(T_-\cap S_+)}\,dx \\
&\qquad + c(\varepsilon)(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \left|V\left(\frac{w}{(1+|D\ell|)\rho}\right)\right|^2 \chi_{(T_-\cap S_+)}\,dx.
\end{aligned}
$$

Compiling these terms, we have

$$
\begin{aligned}
\mathrm{IV} &\le \mathrm{IV}_a + \mathrm{IV}_b + \mathrm{IV}_c + \mathrm{IV}_d \qquad (7.7)\\
&\le \varepsilon (1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat p} \left| V\left(\frac{Du - D\ell}{1+|D\ell|}\right)\right|^2 dx\\
&\qquad + c(\varepsilon)(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \left| V\left(\frac{w}{(1+|D\ell|)\rho}\right)\right|^2 dx.
\end{aligned}
$$

Combining (7.5) and (7.7) gives that for all p_2

$$
\begin{aligned}
\mathrm{IV} &\le \varepsilon (1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \eta^{\hat p} \left| V\left(\frac{Du - D\ell}{1+|D\ell|}\right)\right|^2 dx\\
&\qquad + c(\varepsilon)(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \left| V\left(\frac{w}{(1+|D\ell|)\rho}\right)\right|^2 dx.
\end{aligned}
$$

In the superquadratic case, we estimate term V using **(I)**, Young's inequality with exponent pair $(p_2, \frac{p_2}{p_2-1})$, Corollary 4.3 (i), and Lemma 3.4 (iv)

$$
\begin{aligned}
\mathrm{V} &= \fint_{B_\rho(x_0)} b(x,u,Du)\cdot \eta^{\hat p} w\\
&\le L \fint_{B_\rho(x_0)} \rho (1+|Du|)^{p(x)-1} \eta^{\hat p} \left|\frac{w}{\rho}\right| dx\\
&\le c \fint_{B_\rho(x_0)} \rho^{\frac{p_2}{p_2-1}} (1+|Du|)^{p_2} + \left|\frac{w}{\rho}\right|^{p_2} dx\\
&\le c(1+|D\ell|)^{p_2} \fint_{B_\rho(x_0)} \rho + \left|\frac{w}{(1+|D\ell|)\rho}\right|^{p_2} dx\\
&\le c(1+|D\ell|)^{p_2} \left(\rho + \fint_{B_\rho(x_0)} \left| V\left(\frac{w}{\rho(1+|D\ell|)}\right)\right|^2 dx\right).
\end{aligned}
$$

When p is subquadratic, we consider two distinct cases. On the set T_+ we find the calculations are identical to the superquadratic case. On T_-, we use Corollary 4.3 (i) to find

$$
\begin{aligned}
\mathrm{V} &\le c \fint_{B_\rho(x_0)} \rho (1+|Du|)^{p(x)-1} \left|\frac{w}{\rho}\right| \chi_{(T_-)}\, dx\\
&\le c(1+|D\ell|) \fint_{B_\rho(x_0)} \rho (1+|Du|)^{p_2-1} \left|\frac{w}{\rho(1+|D\ell|)}\right| \chi_{(T_-)}\, dx\\
&\le c(1+|D\ell|) \fint_{B_\rho(x_0)} \rho (1+|Du|)^{p_2-1}\, dx\\
&\le c(1+|D\ell|)^{p_2} \rho,
\end{aligned}
$$

concluding the estimate.

Collecting our terms and choosing ε small enough to be absorbed on the left, we normalise by $(1+|D\ell|)^{p_2}$ to obtain

$$\begin{aligned}
\fint_{B_{\frac{\rho}{2}}(x_0)} \left|V\left(\frac{Du-D\ell}{(1+|D\ell|)}\right)\right|^2 dx &\le \fint_{B_\rho(x_0)} \eta^{\hat{p}} \left|V\left(\frac{Du-D\ell}{(1+|D\ell|)}\right)\right|^2 dx \\
&\le \mathrm{I} = \mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI} \\
&\le c\Bigg(\fint_{B_\rho(x_0)} \left|V\left(\frac{w}{(1+|D\ell|)\rho}\right)\right|^2 dx \\
&\qquad + \omega_\xi^2\left(\fint_{B_\rho(x_0)} |u-\ell(x_0)|\,dx\right) + \mathbf{V}(\rho) + \rho^2\Bigg),
\end{aligned}$$

as required. $\square$

Remark 7.4. *Note if we replace $B_{\frac{\rho}{2}}(x_0)$ with $B_{\theta\rho}(x_0)$ for some $\theta \in (0,1)$, by different choice of cutoff function we can obtain a similar estimate, with our constant now gaining dependence on θ and blowing up as $\theta \to 1$ or $\theta \to 0$.*

$\mathcal{A}$-harmonic approximation

The second step in the proof is to show that the solution to our PDE lies close to a solutions of a family of related linear PDE.

Lemma 7.5 (Approximate $\mathcal{A}$-harmonicity)**.** *Fix $M > 0$ and assume that u is a weak solution to* (7.1) *with* (7.2) *and* (7.3) *under structure conditions* **(V1)**–**(V6)** *with the inhomogeneity satisfying* **(I)***. Then there exists a constant $C = C(M, n, N, L/\nu, \gamma_1, \gamma_2, L_1, L_2, E, \omega_p)$ and a radius $\rho_0 << 1$ such that whenever $\rho < \rho_0$ and $\Phi(x_0, D\ell, \rho) \le \frac{1}{36}$ for some affine map $\ell : \mathbb{R}^n \to \mathbb{R}^N$ satisfying $|D\ell| < M$, there holds*

$$\begin{aligned}
&\left|\fint_{B_\rho(x_0)} \big(D_z a(\,\cdot\,, \ell(x_0), D\ell)\big)_{x_0,\rho}(Du - D\ell)\cdot D\varphi\,dx\right| \\
&\qquad \le c_1(1+|D\ell|)^{p_2-1}\Bigg(\mu\left(\sqrt{\Phi(x_0, D\ell, \rho)}\right)\sqrt{\Phi(x_0, D\ell, \rho)} + \mathbf{V}(\rho) + \mathbf{V}^{\frac{1}{2}}(\rho) + \Phi(x_0, D\ell, \rho) \\
&\qquad\qquad + \omega_\xi\left(\fint_{B_\rho(x_0)} |u-\ell(x_0)|\,dx\right) + \rho\Bigg)\|D\varphi\|_{C(B_\rho(x_0),\mathbb{R}^N)},
\end{aligned}$$

for all $\varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N)$.

Proof of Lemma 7.5: Taking some $\varphi \in C_0^1(B_\rho(x_0), \mathbb{R}^N)$ with $\|D\varphi\|_{L^\infty(B_\rho(x_0),\mathbb{R}^N)} = 1$, we set $v = u - \ell$ and begin by noting

$$\begin{aligned}
&\fint_{B_\rho(x_0)} \big(D_z a(\,\cdot\,, \ell(x_0), D\ell)\big)_{x_0,\rho} Dv \cdot D\varphi \, dx \\
&\quad = \fint_{B_\rho(x_0)} \int_0^1 \Big[\big(D_z a(\,\cdot\,, \ell(x_0), D\ell)\big)_{x_0,\rho} - \big(D_z a(\,\cdot\,, \ell(x_0), D\ell + tDv)\big)_{x_0,\rho}\Big] Dv \cdot D\varphi \, dt\, dx \\
&\qquad\qquad + \fint_{B_\rho(x_0)} \Big[\big(a(\,\cdot\,, \ell(x_0), Du)\big)_{x_0,\rho} - a(x, \ell(x_0), Du)\Big] \cdot D\varphi \, dx \\
&\qquad\qquad + \fint_{B_\rho(x_0)} \big[a(x, \ell(x_0), Du) - a(x, u, Du)\big] \cdot D\varphi \, dx \\
&\qquad\qquad + \fint_{B_\rho(x_0)} b(x, u, Du) \cdot \varphi \, dx \\
&\quad = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},
\end{aligned}$$

with the obvious labelling.

To estimate I we use the differentiability condition **(V5)**, which differs in the super and subquadratic cases. For $2 \le p_2$ we calculate pointwise via **(V5)** and Lemma 3.4 (iv), keeping in mind that $\mu \le 1$

$$\begin{aligned}
&\left| \int_0^1 \Big[\big(D_z a(\,\cdot\,, \ell(x_0), D\ell)\big)_{x_0,\rho} - \big(D_z a(\,\cdot\,, \ell(x_0), D\ell + tDv(x))\big)_{x_0,\rho}\Big] \, dt Dv \cdot D\varphi \right| \\
&\quad \le \int_0^1 \left| \fint_{B_\rho(x_0)} D_z a(\,\cdot\,, \ell(x_0), D\ell) - D_z a(\,\cdot\,, \ell(x_0), D\ell + tDv(x)) \, dy \right| dt |Dv||D\varphi| \\
&\quad \le L \int_0^1 \fint_{B_\rho(x_0)} \left| \mu\left(\frac{|Du(x) - D\ell|}{1 + |D\ell|}\right) \big(1 + |D\ell| + |D\ell + tDv(x)|\big)^{p_2 - 2} \right| dy\, dt |Dv| \\
&\quad \le L \int_0^1 \left| \mu\left(\frac{|Du - D\ell|}{1 + |D\ell|}\right) \big(1 + |D\ell| + |D\ell + tDv(x)|\big)^{p_2 - 2} \right| dt |Du - D\ell| \\
&\quad \le c\mu\left(\frac{|Du - D\ell|}{1 + |D\ell|}\right) \Big[\big(1 + |D\ell|\big)^{p_2 - 2} |Du - D\ell| + |Du - D\ell|^{p_2 - 1}\Big] \\
&\quad \le c\big(1 + |D\ell|\big)^{p_2 - 1} \mu\left(\frac{|Du - D\ell|}{1 + |D\ell|}\right) \left[\left|\frac{Du - D\ell}{1 + |D\ell|}\right| + \left|\frac{Du - D\ell}{1 + |D\ell|}\right|^{p_2 - 1}\right] \\
&\quad \le c\big(1 + |D\ell|\big)^{p_2 - 1} \mu\left(\frac{|Du - D\ell|}{1 + |D\ell|}\right) \left[\left|V\left(\frac{Du - D\ell}{1 + |D\ell|}\right)\right|_{\chi_{S_-}} + \left|V\left(\frac{Du - D\ell}{1 + |D\ell|}\right)\right|^2_{\chi_{S_+}}\right] \\
&\quad \le c\big(1 + |D\ell|\big)^{p_2 - 1} \left[\mu\left(\frac{|Du - D\ell|}{1 + |D\ell|}\right) \left|V\left(\frac{Du - D\ell}{1 + |D\ell|}\right)\right| + \left|V\left(\frac{Du - D\ell}{1 + |D\ell|}\right)\right|^2\right].
\end{aligned}$$

On the other hand, when $1 < p_2 < 2$ we find via **(V5)** and Lemma 3.6

$$
\begin{aligned}
&\left| \int_0^1 \Big[\big(D_z a(\,\cdot\,, \ell(x_0), D\ell)\big)_{x_0,\rho} - \big(D_z a(\,\cdot\,, \ell(x_0), D\ell + tDv(x))\big)_{x_0,\rho} \Big]\, dt Dv \cdot D\varphi \right| \\
&\quad \le \int_0^1 \left| \fint_{B_\rho(x_0)} D_z a(\,\cdot\,, \ell(x_0), D\ell) - D_z a\big(\,\cdot\,, \ell(x_0), D\ell + tDv(x)\big)\, dy \right| dt |Dv||D\varphi| \\
&\quad \le L \int_0^1 \fint_{B_\rho(x_0)} \left| \mu\left(\frac{|Du(x) - D\ell|}{1 + |D\ell|} \right) \left[\frac{1 + |D\ell| + |D\ell + tDv(x)|}{(1 + |D\ell|)(1 + |D\ell + tDv(x)|)} \right]^{2-p_2} \right| dy\, dt |Dv| \\
&\quad \le c \int_0^1 \left| \mu\left(\frac{|Du(x) - D\ell|}{1 + |D\ell|} \right) \left[\frac{1 + |D\ell| + |Dv|}{(1 + |D\ell + tDv|)} \right]^{2-p_2} \right| dt |Dv| \\
&\quad \le c(1 + |D\ell|)^{p_2 - 2} \int_0^1 \left| \mu\left(\frac{|Du(x) - D\ell|}{1 + |D\ell|} \right) \left[\frac{1 + |D\ell + tDv|}{1 + |D\ell| + |Dv|} \right]^{p_2-2} \right| dt |Du - D\ell| \\
&\quad \le c(1 + |D\ell|)^{p_2 - 1} \mu\left(\frac{|Du(x) - D\ell|}{1 + |D\ell|} \right) \left| \frac{Du - D\ell}{1 + |D\ell|} \right|.
\end{aligned}
$$

On S_-, Lemma 3.4 (iv) shows

$$
\begin{aligned}
&c(1 + |D\ell|)^{p_2-1} \mu\left(\frac{|Du - D\ell|}{1 + |D\ell|} \right) \left| \frac{Du - D\ell}{1 + |D\ell|} \right| \\
&\qquad\qquad \le c(1 + |D\ell|)^{p_2-1} \mu\left(\left| V\left(\frac{Du - D\ell}{1 + |D\ell|} \right) \right| \right) \left| V\left(\frac{Du - D\ell}{1 + |D\ell|} \right) \right|,
\end{aligned}
$$

while on S_+ we use the bound $\mu \le 1$ and Lemma 3.4 (iv) to compute

$$
\begin{aligned}
c(1 + |D\ell|)^{p_2-1} \mu\left(\frac{|Du - D\ell|}{1 + |D\ell|} \right) \left| \frac{Du - D\ell}{1 + |D\ell|} \right| &\le c(1 + |D\ell|)^{p_2-1} \left| \frac{Du - D\ell}{1 + |D\ell|} \right|^{p_2} \\
&\le c(1 + |D\ell|)^{p_2-1} \left| V\left(\frac{Du - D\ell}{1 + |D\ell|} \right) \right|^2.
\end{aligned}
$$

Collecting these terms and integrating, then using Hölder's inequality we find

$$
\begin{aligned}
&c(1 + |D\ell|)^{p_2-1} \fint_{B_\rho(x_0)} \left| V\left(\frac{Du - D\ell}{1 + |D\ell|} \right) \right|^2 + \mu\left(\left| V\left(\frac{Du - D\ell}{1 + |D\ell|} \right) \right| \right) \left| V\left(\frac{Du - D\ell}{1 + |D\ell|} \right) \right| dx \\
&\quad \le c(1 + |D\ell|)^{p_2-1} \left[\fint_{B_\rho(x_0)} \left| V\left(\frac{Du - D\ell}{1 + |D\ell|} \right) \right|^2 dx \right. \\
&\qquad\qquad \left. + \left(\fint_{B_\rho(x_0)} \mu^2\left(\left| V\left(\frac{Du - D\ell}{1 + |D\ell|} \right) \right| \right) dx \fint_{B_\rho(x_0)} \left| V\left(\frac{Du - D\ell}{1 + |D\ell|} \right) \right|^2 dx \right)^{\frac{1}{2}} \right].
\end{aligned}
$$

Keeping in mind the concavity of μ^2, we apply Jensen's and Hölder's inequalities to find

$$\left(\fint_{B_\rho(x_0)} \mu^2\left(\left|V\left(\frac{Du - D\ell}{1+|D\ell|}\right)\right|\right) dx\right)^{\frac{1}{2}} \leq \mu\left(\fint_{B_\rho(x_0)} \left|V\left(\frac{Du - D\ell}{1+|D\ell|}\right)\right| dx\right)$$
$$\leq \mu\left(\sqrt{\Phi(x_0, D\ell, \rho)}\right),$$

so we conclude

$$\mathrm{I} \leq c(1+|D\ell|)^{p_2-1}\left(\Phi(x_0, D\ell, \rho) + \mu\left(\sqrt{\Phi(x_0, D\ell, \rho)}\right)\sqrt{\Phi(x_0, D\ell, \rho)}\right).$$

We briefly note that when $|Du| \leq |D\ell|$, we easily have

$$\begin{aligned}
&(1+|D\ell|)^{p_2-1}\mathbf{V}^{\frac{1}{2}}(\rho)\left(\fint_{B_\rho(x_0)} \log^\gamma(1+|Du|)\chi_{(|Du|\leq|D\ell|)}\, dx\right)^{\frac{1}{2}} \\
&\qquad \leq (1+|D\ell|)^{p_2-1}\mathbf{V}^{\frac{1}{2}}(\rho)\left(\fint_{B_\rho(x_0)} \log^\gamma(1+|D\ell|)\, dx\right)^{\frac{1}{2}} \\
&\qquad \leq c(M)(1+|D\ell|)^{p_2-1}\mathbf{V}^{\frac{1}{2}}(\rho).
\end{aligned}$$

On the other hand, when $|D\ell| \leq |Du|$, the inequality $\log(1+|z|) \leq C(\delta)|z|^\delta$, Corollary 4.2, and Corollary 4.3 (i) imply

$$\begin{aligned}
&(1+|D\ell|)^{p_2-1}\mathbf{V}^{\frac{1}{2}}(\rho)\left(\fint_{B_\rho(x_0)} \log^\gamma(1+|Du|)\chi_{(|D\ell|\leq|Du|)}\, dx\right)^{\frac{1}{2}} \\
&\qquad \leq (1+|D\ell|)^{\frac{p_2-2}{2}}\mathbf{V}^{\frac{1}{2}}(\rho)\left(\fint_{B_\rho(x_0)} (1+|Du|)^{p_2}\log^\gamma(1+|Du|)\, dx\right)^{\frac{1}{2}} \\
&\qquad \leq c(\hat{\delta})(1+|D\ell|)^{\frac{p_2-2}{2}}\mathbf{V}^{\frac{1}{2}}(\rho)\left(\fint_{B_\rho(x_0)} (1+|Du|)^{p_2(1+\frac{\hat{\delta}}{4})}\, dx\right)^{\frac{1}{2}} \\
&\qquad \leq c(\hat{\delta})(1+|D\ell|)^{\frac{p_2-2}{2}}\mathbf{V}^{\frac{1}{2}}(\rho)\left(\fint_{B_{2\rho}(x_0)} (1+|Du|)^{p_2}\, dx\right)^{\frac{1+\hat{\delta}}{2}} \\
&\qquad = c(\hat{\delta}, M)(1+|D\ell|)^{p_2-1}\mathbf{V}^{\frac{1}{2}}(\rho),
\end{aligned}$$

and taken together we have

$$(1+|D\ell|)^{p_2-1}\mathbf{V}^{\frac{1}{2}}(\rho)\left(\fint_{B_\rho(x_0)} \log^\gamma(1+|Du|)\chi_{(|D\ell|\leq|Du|)}\, dx\right)^{\frac{1}{2}} \leq c(\hat{\delta}, M)(1+|D\ell|)^{p_2-1}\mathbf{V}^{\frac{1}{2}}(\rho), \tag{7.8}$$

In estimating II we begin by using the VMO condition **(V6)**,

$$
\begin{aligned}
\mathrm{II} &\le \fint_{B_\rho(x_0)} \big|\big(a(\,\cdot\,,\ell(x_0),Du)\big)_{x_0,\rho} - a(x,\ell(x_0),Du)\big||D\varphi|\,dx \\
&\le \fint_{B_\rho(x_0)} \mathbf{v}_{x_0}(x,\rho)(1+|Du|)^{p_2-1}\big[1+\log(1+|Du|)\big]\,dx \\
&\le c\fint_{B_\rho(x_0)} \mathbf{v}_{x_0}(x,\rho)(1+|D\ell|+|Du-D\ell|)^{p_2-1}\big[1+\log(1+|Du|)\big]\,dx \\
&\le c(1+|D\ell|)^{p_2-1}\fint_{B_\rho(x_0)} \left[\mathbf{v}_{x_0}(x,\rho)+\mathbf{v}_{x_0}(x,\rho)\left|\frac{Du-D\ell}{1+|D\ell|}\right|^{p_2-1}\right]\big[1+\log(1+|Du|)\big]\,dx \\
&\le c(1+|D\ell|)^{p_2-1}\fint_{B_\rho(x_0)} \mathbf{v}_{x_0}(x,\rho)\log(1+|Du|)+\big[\mathbf{v}_{x_0}(x,\rho)\log(1+|Du|)\big]^{p_2}\,dx \\
&\qquad + c(1+|D\ell|)^{p_2-1}\fint_{B_\rho(x_0)} \mathbf{v}_{x_0}(x,\rho)+[\mathbf{v}_{x_0}(x,\rho)]^{p_2}+\left|\frac{Du-D\ell}{1+|D\ell|}\right|^{p_2}\,dx.
\end{aligned}
$$

Now since $0\le \mathbf{v}\le 2L$, we have $\mathbf{v}_{x_0}^{p_2}(x,\rho)\le [2L]^{p_2-1}\mathbf{v}_{x_0}(x,\rho)$. Applying Hölder's inequality (with exponents $(2,2)$) and Young's inequalities (with exponent pairs $(2,2)$ and $(p_2,\frac{p_2}{p_2-1})$), (7.8) and Lemma 3.4 (iv), we compute for $2\le p_2$

$$
\begin{aligned}
\mathrm{II} &\le c(1+|D\ell|)^{p_2-1}\fint_{B_\rho(x_0)} \mathbf{v}_{x_0}(x,\rho)+\left|V\!\left(\frac{Du-D\ell}{1+|D\ell|}\right)\right|^2 dx \\
&\qquad + c(1+|D\ell|)^{p_2-1}\fint_{B_\rho(x_0)} \mathbf{v}_{x_0}(x,\rho)\Big[\log(1+|Du|)+\log^{p_2}(1+|Du|)\Big]\,dx \\
&\le c(1+|D\ell|)^{p_2-1}\Bigg(\mathbf{V}(\rho)+\fint_{B_\rho(x_0)}\left|V\!\left(\frac{Du-D\ell}{1+|D\ell|}\right)\right|^2 dx \\
&\qquad + \mathbf{V}^{\frac12}(\rho)\Bigg[\bigg(\fint_{B_\rho(x_0)}\log^2(1+|Du|)\,dx\bigg)^{\frac12}+\bigg(\fint_{B_\rho(x_0)}\log^{2p_2}(1+|Du|)\,dx\bigg)^{\frac12}\Bigg]\Bigg) \\
&\le c(1+|D\ell|)^{p_2-1}\Big(\mathbf{V}(\rho)+\mathbf{V}^{\frac12}(\rho)+\Phi(x_0,D\ell,\rho)\Big).
\end{aligned}
$$

In the subquadratic case, the calculations on S_+ are identical. On S_- we change only the exponent in Young's inequality for a single term from the fourth line above (using the pair $(\frac{2}{p_2-1}, \frac{2}{3-p_2}))$ to compute

$$
\begin{aligned}
(1{+}|D\ell|)^{p_2-1} &\fint_{B_\rho(x_0)} \mathbf{v}_{x_0}(x,\rho) \Big|\frac{Du-D\ell}{1+|D\ell|}\Big|^{p_2-1} \big[1+\log(1+|Du|)\big]\,dx \\
&\le (1+|D\ell|)^{p_2-1} \fint_{B_\rho(x_0)} \mathbf{v}_{x_0}^{\frac{2}{3-p_2}}(x,\rho)\big[1+\log(1+|Du|)\big]^{\frac{2}{3-p_2}} + \Big|\frac{Du-D\ell}{1+|D\ell|}\Big|^2\,dx \\
&\le c(1+|D\ell|)^{p_2-1} \fint_{B_\rho(x_0)} \mathbf{v}_{x_0}(x,\rho)\Big[1+\log^{\frac{2}{3-p_2}}(1+|Du|)\Big] + \Big|\frac{Du-D\ell}{1+|D\ell|}\Big|^2\,dx \\
&\le c(1+|D\ell|)^{p_2-1}\bigg(\mathbf{V}(\rho)+\Phi(x_0,D\ell,\rho)+\mathbf{V}^{\frac12}(\rho)\Big(\fint_{B_\rho(x_0)} \log^{\frac{4}{3-p_2}}(1+|Du|^2)\,dx\Big)^{\frac12}\bigg) \\
&\le c(1+|D\ell|)^{p_2-1}\Big(\mathbf{V}(\rho)+\Phi(x_0,D\ell,\rho)+\mathbf{V}^{\frac12}(\rho)\Big).
\end{aligned}
$$

Owing to **(V4)** we have for III and all $p_2>1$

$$
\begin{aligned}
\mathrm{III} &\le \fint_{B_\rho(x_0)} \big|a(x,\ell(x_0),Du)-a(x,u,Du)\big|\,dx \\
&\le c\fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big)(1+|Du|)^{p(x)-1}\,dx \\
&\le c\fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big)(1+|Du-D\ell|+|D\ell|)^{p_2-1}\,dx \\
&\le c\fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big)(1+|D\ell|)^{p_2-1} + \omega_\xi\big(|u-(u)_{x_0,\rho}|\big)|Du-D\ell|^{p_2-1}\,dx \\
&\le c\,(1+|D\ell|)^{p_2-1}\fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big)\bigg[1+\Big|\frac{Du-D\ell}{1+|D\ell|}\Big|^{p_2-1}\bigg]\,dx.
\end{aligned}
$$

Now when $p\ge 2$ we use Young's inequality (with exponent pair $(p_2,\frac{p_2}{p_2-1})$), the fact $\omega_\xi\le 1$, Lemma 3.4 (iv), then Jensen's inequality to calculate

$$
\begin{aligned}
&\fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big)\bigg[1+\Big|\frac{Du-D\ell}{1+|D\ell|}\Big|^{p_2-1}\bigg]\,dx \qquad (7.9)\\
&\qquad\le c\fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big)+\omega_M^{p_2}\big(|u-\ell(x_0)|\big)+\Big|\frac{Du-D\ell}{1+|D\ell|}\Big|^{p_2}\,dx \\
&\qquad\le c\fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big)+\Big|V\Big(\frac{Du-D\ell}{1+|D\ell|}\Big)\Big|^2\,dx \\
&\qquad\le c\bigg[\omega_\xi\Big(\fint_{B_\rho(x_0)} |u-\ell(x_0)|\,dx\Big)+\fint_{B_\rho(x_0)}\Big|V\Big(\frac{Du-D\ell}{1+|D\ell|}\Big)\Big|^2\,dx\bigg].
\end{aligned}
$$

Considering the case where $1<p_2<2$, the estimates on S_+ are analogous to the superquadratic case, and on S_- we change only the exponents in Young's inequality to

$(\frac{2}{p_2-1}, \frac{2}{3-p_2})$, to deduce

$$\begin{aligned}
&\fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big)\left[1+\left|\frac{Du-D\ell}{1+|D\ell|}\right|^{p_2-1}\right]\chi_{S_-}\,dx \qquad (7.10)\\
&\qquad \le c\fint_{B_\rho(x_0)} \omega_\xi\big(|u-\ell(x_0)|\big)+\omega_M^{\frac{2}{3-p_2}}\big(|u-\ell(x_0)|\big)+\left|\frac{Du-D\ell}{1+|D\ell|}\right|^2\chi_{S_-}\,dx\\
&\qquad \le c\left[\omega_\xi\bigg(\fint_{B_\rho(x_0)}\big|u-\ell(x_0)\big|\,dx\bigg)+\fint_{B_\rho(x_0)}\left|V\bigg(\frac{Du-D\ell}{1+|D\ell|}\bigg)\right|^2dx\right].
\end{aligned}$$

By comparing (7.9) to (7.10), we see that for any $p_2>1$ we have

$$\mathrm{III}\le c(1+|Du|)^{p_2-1}\left[\omega_\xi\bigg(\fint_{B_\rho(x_0)}\big|u-\ell(x_0)\big|\,dx\bigg)+\fint_{B_\rho(x_0)}\left|V\bigg(\frac{Du-D\ell}{1+|D\ell|}\bigg)\right|^2dx\right].$$

When the inhomogeneity b satisfies the controllable growth condition **(I)**, we can estimate via Corollary 4.3 (i)

$$\begin{aligned}
\mathrm{IV}&\le\fint_{B_\rho(x_0)}\big|b(x,u,Du)\cdot\varphi\big|\,dx\\
&\le L\fint_{B_\rho(x_0)}(1+|Du|)^{p(x)-1}\rho\,dx\\
&\le L\fint_{B_\rho(x_0)}(1+|Du|)^{p_2-1}\rho\,dx\\
&\le c(1+|D\ell|)^{p_2-1}\rho.
\end{aligned}$$

Assembling our terms, we have in either case after possibly restricting ρ_0 to ensure that $\mathbf{V}(\rho)<1$

$$\begin{aligned}
&\left|\fint_{B_\rho(x_0)}\big(D_za(\,\cdot\,,(u)_{x_0,\rho},D\ell)\big)_{x_0,\rho}Dv\cdot D\varphi\,dx\right|\\
&\qquad\le c_1(1+|D\ell|)^{p_2-1}\Bigg(\mu\Big(\sqrt{\Phi(x_0,D\ell,\rho)}\Big)\sqrt{\Phi(x_0,D\ell,\rho)}+\mathbf{V}^{\frac12}(\rho)+\Phi(x_0,D\ell,\rho)\\
&\qquad\qquad+\omega_\xi\bigg(\fint_{B_\rho(x_0)}\big|u-\ell(x_0)\big|\,dx\bigg)+\rho\Bigg)\|D\varphi\|_{C(B_\rho(x_0),\mathbb{R}^N)}.
\end{aligned}$$

This shows the claim for test functions satisfying $\|D\varphi\|_{L^\infty(B_\rho(x_0),\mathbb{R}^N)}=1$, the full result follows via rescaling of the test function. $\square$

Application of the $\mathcal{A}$-harmonic approximation lemma

We now recall

$$\Psi(x_0,\ell,\rho) := \fint_{B_\rho(x_0)} \left| V\left(\frac{u-\ell}{\rho(1+|D\ell|)}\right)\right|^2 dx,$$
$$\Phi(x_0,D\ell,\rho) := \fint_{B_\rho(x_0)} \left| V\left(\frac{Du-D\ell}{1+|D\ell|}\right)\right|^2 dx,$$

and write

$$\Upsilon(x_0,\ell,\rho) := \omega_\xi\bigg(\fint_{B_\rho(x_0)} \big|u-\ell(x_0)\big|\, dx\bigg),$$
$$M(x_0,\rho) := \rho \fint_{B_\rho(x_0)} |Du|\, dx.$$

We further define

$$E(x_0,\ell,\rho) := \Psi(x_0,\ell,\rho) + \mathbf{V}^{\frac{1}{2}}(\rho) + \Upsilon(x_0,\ell,\rho) + \omega_p^{\frac{1}{2}}(\rho) + \rho.$$

In view of this notation, we can estimate the Caccioppoli inequality from Lemma 7.3 by

$$\Phi\Big(x_0,D\ell,\frac{\rho}{2}\Big) \le c_c E(x_0,\ell,\rho). \tag{7.11}$$

Plugging (7.11) into Lemma 7.5 on $B_{\frac{\rho}{2}}(x_0)$ we deduce that provided $\Phi\left(x_0,D\ell,\frac{\rho}{2}\right) \le \frac{1}{36}$, there holds

$$\begin{aligned}
&\bigg|\fint_{B_{\frac{\rho}{2}}(x_0)} \big(D_z a(\,\cdot\,,\ell(x_0),D\ell)\big)_{x_0,\frac{\rho}{2}} Dv\cdot D\varphi\, dx\bigg| \\
&\quad\le c_1(1+|D\ell|)^{p_2-1}\Big[\Phi\big(x_0,D\ell,\rho/2\big) + \mu\Big(\sqrt{\Phi(x_0,D\ell,\rho/2)}\Big)\sqrt{\Phi(x_0,D\ell,\rho/2)} \\
&\qquad\qquad + \mathbf{V}^{\frac{1}{2}}\big(\rho/2\big) + \Upsilon^{\frac{1}{2}}(x_0,\ell,\rho) + \rho\Big] \\
&\quad\le c_1 c_c(1+|D\ell|)^{p_2-1}\left(E(x_0,\ell,\rho) + \mu\left(\sqrt{E\left(x_0,\ell,\rho\right)}\right)\sqrt{E\left(x_0,\ell,\rho\right)}\right) \\
&\quad\le c_a(1+|D\ell|)^{p_2-1}\left[\sqrt{E(x_0,\ell,\rho)} + \mu\left(\sqrt{E\left(x_0,\ell,\rho\right)}\right)\right]\sqrt{E\left(x_0,\ell,\rho\right)},
\end{aligned}$$

where we have relabelled the constant.

Having established this preliminary estimate, we can fix these excesses small enough to invoke the $\mathcal{A}$-harmonic approximation lemma. The a priori bounds on the solution to the linearised PDE, combined with our Caccioppoli inequality, allow us to demonstrate a preliminary rescaling estimate on the Campanato style excess functional Φ. This estimate is then iterated, and finally an interpolation argument is provided to reproduce the estimate at all scales.

$\mathcal{A}$-harmonic approximation

For δ_0 to be chosen later we now restrict ρ to be small enough to ensure

$$\kappa := \sqrt{E\big(x_0, \ell_{x_0,\rho}, \rho\big)} \le 1 \qquad \text{and} \qquad \sqrt{E\big(x_0, \ell_{x_0,\rho}, \rho\big)} + \mu\Big(\sqrt{E\big(x_0, \ell_{x_0,\rho}, \rho\big)}\Big) \le \delta_0. \tag{7.12}$$

We now set $\ell = \ell_{x_0,\frac{\rho}{2}}$ so Lemma 7.5 is satisfied with

$$\mathcal{A} := \frac{\big(D_z a(\,\cdot\,, (u)_{B_{\frac{\rho}{2}}(x_0)}, D\ell_{x_0,\frac{\rho}{2}})\big)_{x_0,\frac{\rho}{2}}}{c_a(1+|D\ell_{x_0,\frac{\rho}{2}}|)^{p_2-2}} \qquad \text{and} \qquad w := \frac{u-\ell_{x_0,\frac{\rho}{2}}}{1+|D\ell_{x_0,\frac{\rho}{2}}|}.$$

Given any $\varepsilon > 0$, this implies via Lemma 3.12 the existence of an $\mathcal{A}-$harmonic function h satisfying the a priori estimates

$$\sup_{B_{\frac{\rho}{2}}(x_0)} \left(2\left|\frac{h}{\rho}\right| + |Dh| + \frac{\rho}{2}|D^2h|\right) \le c_h \qquad \text{and} \qquad \fint_{B_{\frac{\rho}{2}}(x_0)} \left|V\left(\frac{w-\kappa h}{\rho/2}\right)\right|^2 dx \le \kappa^2\varepsilon, \tag{7.13}$$

provided δ_0 is small enough.

Preliminary decay estimate

Note that via Taylor's theorem and (7.13) we immediately have

$$\sup_{B_{\theta\rho}(x_0)} |h(x) - h(x_0) - Dh(x_0)(x-x_0)| \le \sup_{B_\rho(x_0)} |D^2h(\theta\rho)^2| \le c_h\theta^2\rho \tag{7.14}$$

for any $x \in B_{\theta\rho}(x_0)$ where $\theta \in (0, \frac{1}{4})$.

We further impose the smallness condition $\kappa \le c_h^{-1}$ on (7.12), which ensures

$$\kappa\frac{|h(x) - h(x_0) - Dh(x_0)(x-x_0)|}{\theta\rho} \le c_h\kappa\theta < 1,$$

and so by Lemma 3.4 (iv) we have

$$\fint_{B_{\theta\rho}(x_0)} \left|V\left(\kappa\frac{h-h(x_0)-Dh(x_0)(x-x_0)}{\theta\rho}\right)\right|^2 dx \le c_h^2\theta^2\kappa^2. \tag{7.15}$$

Lemma 7.6. *For every $M > 2$ there exist constants $0 < \hat{\rho}, \theta < \frac{1}{4}$ such that whenever*

$\Phi(x_0, D\ell, \theta\rho) \le \frac{1}{36}$ and $|D\ell| \le M$ and the smallness conditions

$$\Psi(x_0, \ell_{x_0,\rho}, \rho) < \left(\frac{1}{4}\frac{\theta^{n+1}}{n+2}\right)^2, \qquad E(x_0, \ell_{x_0,\rho}, \rho) \le \frac{1}{c_h^2}, \qquad \varepsilon \le \theta^{n+2+\max\{2,p_2\}},$$

$$\rho < \hat{\rho} \qquad \text{and} \qquad \sqrt{E(x_0, \ell_{x_0,\rho}, \rho)} + \mu\Big(\sqrt{E(x_0, \ell_{x_0,\rho}, \rho)}\Big) \le \delta_0 \tag{7.16}$$

hold, then for all $k \in \mathbb{N}$ we have

$$\Psi(x_0, \ell_{x_0,\theta\rho}, \theta\rho) \le c_d \theta^2 E\left(x_0, \ell_{x_0,\rho}, \rho\right).$$

Proof of Lemma 7.6: Using Corollary 3.11, then Lemma 3.10 with $\lambda = \frac{1}{\theta\rho(1+|D\ell_{x_0,\frac{\rho}{2}}|)}$, Lemma 3.4 (ii) then (7.13) and (7.15), we write $\hat{p} = \max\{2, p_2\}$ and calculate

$$\begin{aligned}
\Psi(x_0, \ell_{x_0,\theta\rho}, \theta\rho) &= \fint_{B_{\theta\rho}(x_0)} \left|V\left(\frac{u - \ell_{x_0,\theta\rho}}{\theta\rho(1 + |\ell_{x_0,\theta\rho}|)}\right)\right|^2 dx \\
&\le 2^{\hat{p}} \fint_{B_{\theta\rho}(x_0)} \left|V\left(\frac{u - \ell_{x_0,\theta\rho}}{\theta\rho(1 + |\ell_{x_0,\frac{\rho}{2}}|)}\right)\right|^2 dx \\
&\le c(n, p_2) \fint_{B_{\theta\rho}(x_0)} \left|V\left(\frac{w - \kappa(h(x_0) + Dh(x_0)(x - x_0)}{\theta\rho})\right)\right|^2 dx \\
&\le c(n, p_2)\theta^{-n-\hat{p}} \fint_{B_{\frac{\rho}{2}}(x_0)} \left|V\left(\frac{w - \kappa h}{\rho/2}\right)\right|^2 dx \\
&\qquad + c(n, p_2) \fint_{B_{\theta\rho}(x_0)} \left|V\left(\kappa\frac{h - h(x_0) - Dh(x_0)(x - x_0)}{\theta\rho}\right)\right|^2 dx \\
&\le c(n, p_2)\theta^{-n-\hat{p}}\kappa^2\varepsilon + 2^{2p_2} c_h^2 \theta^2 \kappa^2.
\end{aligned}$$

Choosing $\varepsilon = \theta^{n+2+\hat{p}}$ and keeping in mind our definition of κ, this is simply

$$\Psi(x_0, \ell_{x_0,\theta\rho}, \theta\rho) \le c\theta^2 E(x_0, \ell_{x_0,\frac{\rho}{2}}, \rho). \tag{7.17}$$

To show

$$E(x_0, \ell_{x_0,\frac{\rho}{2}}, \rho) \le cE(x_0, \ell_{x_0,\rho}, \rho),$$

we first consider the term Ψ. By Lemma 3.4 (i) and (ii), together with Corollary 3.11

we have

$$\Psi(x_0,\ell_{x_0,\frac{\rho}{2}},\rho) = \fint_{B_\rho(x_0)} \left|V\left(\frac{u-\ell_{x_0,\frac{\rho}{2}}}{\rho(1+|D\ell_{x_0,\frac{\rho}{2}}|)}\right)\right|^2 dx \tag{7.18}$$
$$\leq \left[\frac{1+|D\ell_{x_0,\rho}|}{1+|D\ell_{x_0,\frac{\rho}{2}}|}\right]^{\hat{p}} \fint_{B_\rho(x_0)} \left|V\left(\frac{u-\ell_{x_0,\frac{\rho}{2}}}{\rho(1+|D\ell_{x_0,\rho}|)}\right)\right|^2 dx$$
$$\leq c(p_2)\fint_{B_\rho(x_0)} \left|V\left(\frac{u-\ell_{x_0,\rho}}{\rho(1+|D\ell_{x_0,\rho}|)}\right)\right|^2 + \left|V\left(\frac{\ell_{x_0,\frac{\rho}{2}}-\ell_{x_0,\rho}}{\rho(1+|D\ell_{x_0,\rho}|)}\right)\right|^2 dx.$$

Estimate (3.6) from Lemma 3.9 and the fact that $\fint_{B_{\frac{\rho}{2}}(x_0)} D\ell(x-x_0)\,dx = 0$ give us the pointwise estimate

$$\left|\ell_{x_0,\frac{\rho}{2}} - \ell_{x_0,\rho}\right| \leq \left|\ell_{x_0,\rho}(x_0) - \ell_{x_0,\frac{\rho}{2}}(x_0)\right| + \rho\left|D\ell_{x_0,\frac{\rho}{2}} - D\ell_{x_0,\rho}\right| \tag{7.19}$$
$$\leq \left|\fint_{B_{\frac{\rho}{2}}(x_0)} u - \ell_{x_0,\rho}(x_0)\,dx\right| + 2(n+2)\fint_{B_{\frac{\rho}{2}}(x_0)} \left|u-\ell_{x_0,\rho}\right| dy$$
$$\leq 2^{n+2}(n+3)\fint_{B_\rho(x_0)} \left|u-\ell_{x_0,\rho}\right| dy.$$

By the almost-convexity of V given in (3.2) we calculate via Jensen's inequality

$$\fint_{B_\rho(x_0)} \left|V\left(\frac{\ell_{x_0,\frac{\rho}{2}}-\ell_{x_0,\rho}}{\rho(1+|\ell_{x_0,\rho}|)}\right)\right|^2 dx \leq \fint_{B_\rho(x_0)} \left|V\left(\frac{2^{n+2}(n+3)\fint_{B_\rho(x_0)}|u(y)-\ell_{x_0,\rho}(y)|\,dy}{\rho(1+|\ell_{x_0,\rho}|)}\right)\right|^2 dx$$
$$\leq c\fint_{B_\rho(x_0)} \left|V\left(\frac{u-\ell_{x_0,\rho}}{\rho(1+|\ell_{x_0,\rho}|)}\right)\right|^2 dx. \tag{7.20}$$

Combining (7.18) and (7.20), we have

$$\Psi(x_0,\ell_{x_0,\frac{\rho}{2}},\rho) \leq c(n,p_2)\Psi(x_0,\ell_{x_0,\rho},\rho),$$

provided

$$\Psi(x_0,\ell_{x_0,\rho},\rho) \leq \frac{1}{4}\left(\frac{\theta^{n+1}}{n+2}\right)^2.$$

We now show $\Upsilon(x_0,\ell_{x_0,\frac{\rho}{2}},\rho) \leq c\,\Upsilon(x_0,\ell_{x_0,\rho},\rho)$, which will follow from the concavity (and hence subadditivity) of ω_ξ, once we compute as in (7.19)

$$\fint_{B_\rho(x_0)} |u-\ell_{x_0,\frac{\rho}{2}}(x_0)|\,dx \leq \fint_{B_\rho(x_0)} |u-(u)_{x_0,\rho}|\,dx + |(u)_{B_{\frac{\rho}{2}}(x_0)} - (u)_{x_0,\rho}|$$
$$\leq 2^{n+1}\fint_{B_\rho(x_0)} |u-(u)_{x_0,\rho}|\,dx,$$

and since ω_ξ is concave we have

$$\Upsilon(x_0, \ell_{x_0,\frac{\rho}{2}}, \rho) \leq 2^{n+1}\Upsilon(x_0, \ell_{x_0,\rho}, \rho).$$

Collecting terms, and noting that the other terms in $E\left(x_0, \ell_{x_0,\rho}, \rho\right)$ are monotone in ρ, we have shown whenever

$$\Psi(x_0, \ell_{x_0,\rho}, \rho) \leq \left(\frac{1}{4}\frac{\theta^{n+1}}{n+2}\right)^2, \qquad \text{there holds} \qquad E(x_0, \ell_{x_0,\frac{\rho}{2}}, \rho) \leq c(n, p_2)E(x_0, \ell_{x_0,\rho}, \rho).$$

Plugging this into (7.17) we conclude

$$\Psi(x_0, \ell_{x_0,\theta\rho}, \theta\rho) \leq c_d\theta^2 E(x_0, \ell_{x_0,\rho}, \rho), \tag{7.21}$$

which is the desired estimate. □

Proof of Theorem 7.1

Choice of constants

We now take $\gamma < n$ to be fixed later, and set

$$\begin{aligned} \iota &= \min\left\{\tfrac{1}{6c_c}\left(\tfrac{\theta^{n+1}}{4(n+2)}\right)^{\max\{2,p_2\}}\beta, \left(\tfrac{1}{4}\tfrac{\theta^{n+1}}{n+2}\right)^2\right\}, &\qquad \beta &= \left(\tfrac{\theta^n}{2c(p)}\right)^{\max\{2,p_2\}}, \\ \theta &= \min\left\{\left(\tfrac{1}{5c_d}\right)^{\frac{1}{2}}, \tfrac{1}{8}, \left(\tfrac{1}{2}\right)^{\frac{1}{n-\gamma}}\right\}, &\qquad \hat{\rho} &\leq \min\left\{\tfrac{\rho_0}{2}, \sigma, \iota\right\}, \end{aligned} \tag{7.22}$$

and ensure σ is small enough to satisfy

$$\omega_\xi(\sigma), \mathbf{V}^{\frac{1}{2}}(\sigma), \omega_p(\sigma) \leq \iota. \tag{7.23}$$

Note all these constants depend only on $n, N, M, \gamma_1, \gamma_2, L/\nu, L_1, L_2, E, \omega_p$, and μ.

We can now iterate this procedure to show the following:

Almost BMO estimate

Lemma 7.7. *For every $M > 2$ there exist constants $0 < \iota, \beta, \sigma, \hat{\rho}, \theta < 1$ satisfying (7.22) and (7.23), such that whenever $\Phi(x_0, \ell, \theta^k\rho) \leq \frac{1}{36}$, $|D\ell| \leq M$, and the smallness conditions*

$$\begin{aligned} \Psi(x_0, \ell_{x_0,\rho}, \rho) < \iota, \qquad &\Phi(x_0, D\ell_{x_0,\rho}, \rho) < \beta, \qquad M(x_0, \rho) < \sigma, \\ \textit{and} \qquad &\rho < \hat{\rho} \end{aligned} \tag{7.24}$$

hold, then for all $k \in \mathbb{N}$ we have

$$\Psi(x_0, \ell_{x_0,\theta^k\rho}, \theta^k\rho) < \iota, \qquad \Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^k\rho) < \beta, \qquad \textit{and} \qquad M(x_0, \theta^k\rho) < \sigma.$$

Proof of Lemma 7.7: In estimating $M(x_0, \theta^k\rho)$ we assume only that $M(x_0, \rho) < \sigma$ and $\Phi(x_0, D\ell_{x_0,\theta\rho}, \theta\rho) < \iota$. By the principle of induction it suffices to show that $M(x_0, \theta^k\rho) < \sigma$ and $\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^k\rho) < \iota$ imply $M(x_0, \theta^{k+1}\rho) < \sigma$.

We begin by calculating

$$\begin{aligned} M(x_0, \theta^{k+1}\rho) &= \theta^{k+1}\rho \fint_{B_{\theta^{k+1}\rho}(x_0)} |Du| \, dx \\ &\leq \theta^{k+1}\rho \fint_{B_{\theta^{k+1}\rho}(x_0)} \left|Du - (Du)_{x_0,\theta^k\rho}\right| dx + \theta^{k+1}\rho \left| (Du)_{x_0,\theta^k\rho} \right| \\ &\leq \theta^{1-n}\theta^k\rho \fint_{B_{\theta^k\rho}(x_0)} \left|Du - (Du)_{x_0,\theta^k\rho}\right| dx + \theta M(x_0, \theta^k\rho). \end{aligned} \tag{7.25}$$

Again writing $\hat{p} = \max\{2, p_2\}$, we note that Corollary 4.3 (ii) together with Lemma 3.4 (iv) and Hölder's inequality let us calculate

$$\begin{aligned} \fint_{B_{\theta^k\rho}(x_0)} \frac{|Du - (Du)_{x_0,\theta^k\rho}|}{1 + |(Du)_{x_0,\theta^k\rho}|} \, dx &\leq \frac{1 + |D\ell_{x_0,\theta^k\rho}|}{1 + |(Du)_{x_0,\theta^k\rho}|} \fint_{B_{\theta^k\rho}(x_0)} \frac{|Du - (Du)_{x_0,\theta^k\rho}|}{1 + |D\ell_{x_0,\theta^k\rho}|} \, dx \\ &\leq 2 \fint_{B_{\theta^k\rho}(x_0)} \frac{|Du - (Du)_{x_0,\theta^k\rho}|}{1 + |D\ell_{x_0,\theta^k\rho}|} \, dx \\ &\leq 4 \left(\fint_{B_{\theta^k\rho}(x_0)} \left| V\left(\frac{Du - D\ell_{x_0,\theta^k\rho}}{1 + |D\ell_{x_0,\theta^k\rho}|} \right) \right|^2 dx \right)^{\frac{1}{\hat{p}}} \\ &\leq 4c(p_2) \Big(\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^k\rho) \Big)^{\frac{1}{\hat{p}}} \\ &=: 4c_m \Big(\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^k\rho) \Big)^{\frac{1}{\hat{p}}}. \end{aligned}$$

In order to estimate the first term on the last line of (7.25), we can first compute

$$\begin{aligned} \theta^k\rho \fint_{B_{\theta^k\rho}(x_0)} \left|Du - (Du)_{x_0,\theta^k\rho}\right| dx &= \theta^k\rho \big(1 + |(Du)_{x_0,\theta^k\rho}|\big) \fint_{B_{\theta^k\rho}(x_0)} \frac{|Du - (Du)_{x_0,\theta^k\rho}|}{1 + |(Du)_{x_0,\theta^k\rho}|} \, dx \\ &\leq 4c_m\theta^k\rho \Big(\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^k\rho) \Big)^{\frac{1}{\hat{p}}} \left(1 + \fint_{B_{\theta^k\rho}(x_0)} |Du| \, dx \right) \\ &= 4c_m \Big(\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^k\rho) \Big)^{\frac{1}{\hat{p}}} M(x_0, \theta^k\rho) \\ &\qquad + 4c_m\theta^k\rho \Big(\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^k\rho) \Big)^{\frac{1}{\hat{p}}}. \end{aligned}$$

Hence, via (7.25) we see

$$\begin{aligned}
M(x_0, \theta^{k+1}\rho) &\le \theta^{1-n}\theta^k\rho \fint_{B_{\theta^k\rho}(x_0)} \left|Du - (Du)_{x_0,\theta^k\rho}\right| dx + \theta M(x_0, \theta^k\rho) \\
&\le 4c_m\theta^{1-n}\Big(\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^k\rho)\Big)^{\frac{1}{\hat{p}}} M(x_0, \theta^k\rho) \\
&\qquad + 4c_m\theta^{1-n}\theta^k\rho\Big(\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^k\rho)\Big)^{\frac{1}{\hat{p}}} \\
&\qquad + \theta M(x_0, \theta^k\rho) \\
&\le \frac{M(x_0, \theta^k\rho)}{4} + \frac{\theta^k\rho}{4} + \frac{M(x_0, \theta^k\rho)}{8} \\
&\le \sigma,
\end{aligned}$$

whenever

$$\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^k\rho) \le \left(\frac{\theta^n}{16c_m}\right)^{\hat{p}}, \qquad \text{and} \qquad \rho \le \frac{\sigma}{\theta^k},$$

which holds by (7.22). Since $\hat{\rho} \le \sigma$ satisfies

$$\mathbf{V}(\hat{\rho}) < \iota^2, \qquad \omega_p(\hat{\rho}) < \iota, \qquad \hat{\rho} < \iota,$$

and σ is small enough to ensure

$$\omega_\xi(\sigma) \le \iota,$$

we can use estimate (7.21) with $\theta^k\rho$ in place of ρ to establish

$$\begin{aligned}
\Psi(x_0, \ell_{x_0,\theta^{k+1}\rho}, \theta^{k+1}\rho) &\le c_d\theta^2 E(x_0, \ell_{x_0,\theta^k\rho}, \theta^k\rho) \\
&= c_d\theta^2\Big(\Psi(x_0, \ell_{x_0,\theta^k\rho}, \theta^k\rho) + \mathbf{V}^{\frac{1}{2}}(\theta^k\rho) \\
&\qquad + \omega_p(\theta^k\rho) + \theta^k\rho + \Upsilon(x_0, \ell_{x_0,\theta^k\rho}, \theta^k\rho)\Big) \\
&< c_d\theta^2\Big(\iota + \mathbf{V}^{\frac{1}{2}}(\hat{\rho}) + \omega_p(\hat{\rho}) + \hat{\rho} + \omega_\xi\big(M(x_0, \theta^k\rho)\big)\Big) \\
&\le 5c_d\theta^2\iota \\
&\le \iota,
\end{aligned}$$

provided $\theta \le \left(\frac{1}{6c_d}\right)^{\frac{1}{2}}$, which holds by (7.22). Finally we show the estimate for Φ. Note first of all that via Remark 7.4 we can calculate

$$\begin{aligned}
\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^{k+1}\rho) &\le c_c E(x_0, \ell_{x_0,\theta^k\rho}, \theta^k\rho) \\
&\le 6c_c\iota,
\end{aligned} \tag{7.26}$$

so it remains to estimate $\Phi(x_0, D\ell_{x_0,\theta^{k+1}\rho}, \theta^{k+1}\rho)$ in terms of $\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^{k+1}\rho)$. We begin by noting Lemma 3.4 (i) and (ii) with Corollary 3.11 together yield

$$\begin{aligned}\Phi(x_0, D\ell_{x_0,\theta^{k+1}\rho}, \theta^{k+1}\rho) &= \fint_{B_{\theta^{k+1}\rho}(x_0)} \left| V\left(\frac{Du - D\ell_{x_0,\theta^{k+1}\rho}}{1+|D\ell_{x_0,\theta^{k+1}\rho}|}\right)\right|^2 dx \\ &\leq \fint_{B_{\theta^{k+1}\rho}(x_0)} \left| V\left(\frac{1+|D\ell_{x_0,\theta^k\rho}|}{1+|D\ell_{x_0,\theta^{k+1}\rho}|}\frac{Du - D\ell_{x_0,\theta^{k+1}\rho}}{1+|D\ell_{x_0,\theta^k\rho}|}\right)\right|^2 dx \\ &\leq 2^{\hat{p}} \fint_{B_{\theta^{k+1}\rho}(x_0)} \left| V\left(\frac{Du - D\ell_{x_0,\theta^{k+1}\rho}}{1+|D\ell_{x_0,\theta^k\rho}|}\right)\right|^2 dx \\ &\leq 2^{p_2+\hat{p}}\left[\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^{k+1}\rho) + \left| V\left(\frac{D\ell_{x_0,\theta^k\rho} - D\ell_{x_0,\theta^{k+1}\rho}}{1+|D\ell_{x_0,\theta^k\rho}|}\right)\right|^2\right].\end{aligned} \tag{7.27}$$

When considering the second term, we begin by noting via Lemma 3.9

$$\begin{aligned}\left|D\ell_{x_0,\theta^k\rho} - D\ell_{x_0,\theta^{k+1}\rho}\right| &\leq \left(\frac{n+2}{\theta^{k+1}\rho}\right)\fint_{B_{\theta^{k+1}\rho}(x_0)} \left|u - \ell_{x_0,\theta^k\rho}\right| dx \\ &\leq \left(\frac{n+2}{\theta^{n+1}}\right)\fint_{B_{\theta^k\rho}(x_0)} \left|\frac{u - \ell_{x_0,\theta^k\rho}}{\theta^k\rho}\right| dx.\end{aligned}$$

So Lemma 3.4 (i), the almost-convexity of V as per (3.2), and Jensen's inequality imply

$$\begin{aligned}\left| V\left(\frac{D\ell_{x_0,\theta^k\rho} - D\ell_{x_0,\theta^{k+1}\rho}}{1+|D\ell_{x_0,\theta^k\rho}|}\right)\right|^2 &\leq \left| V\left(\frac{n+2}{\theta^{n+1}}\fint_{B_{\theta^k\rho}(x_0)} \left|\frac{u - \ell_{x_0,\theta^k\rho}}{\theta^k\rho(1+|D\ell_{x_0,\theta^k\rho}|)}\right| dx\right)\right|^2 \\ &\leq \left(\frac{n+2}{\theta^{n+1}}\right)^{\max\{2,p_2\}}\left| V\left(\fint_{B_{\theta^k\rho}(x_0)} \left|\frac{u - \ell_{x_0,\theta^k\rho}}{\theta^k\rho(1+|D\ell_{x_0,\theta^k\rho}|)}\right| dx\right)\right|^2 \\ &\leq \left(\frac{n+2}{\theta^{n+1}}\right)^{\max\{2,p_2\}}\fint_{B_{\theta^k\rho}(x_0)} \left| V\left(\left|\frac{u - \ell_{x_0,\theta^k\rho}}{\theta^k\rho(1+|D\ell_{x_0,\theta^k\rho}|)}\right|\right)\right|^2 dx.\end{aligned}$$

Plugging this into (7.27), in view of (7.26) and

$$\iota \leq \frac{1}{6c_c}\left(\frac{\theta^{n+1}}{4(n+2)}\right)^{\max\{2,p_2\}}\beta,$$

from (7.22), we find

$$\begin{aligned}\Phi(x_0, D\ell_{x_0,\theta^{k+1}\rho}, \theta^{k+1}\rho) &\leq 2^{p_2+\max\{2,p_2\}}\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^{k+1}\rho) \\ &\qquad + 2^{p_2}\left(2\frac{n+2}{\theta^{n+1}}\right)^{\max\{2,p_2\}}\Phi(x_0, D\ell_{x_0,\theta^k\rho}, \theta^k\rho) \\ &\leq 2^{p_2+\max\{2,p_2\}}\left[6c_c + \left(\frac{n+2}{\theta^{n+1}}\right)^{\max\{2,p_2\}}\right]\iota \\ &< 6c_c\left(4\frac{n+2}{\theta^{n+1}}\right)^{\max\{2,p_2\}}\iota \\ &\leq \beta.\end{aligned}$$

□

Iteration

We proceed to calculate via Corollary 4.3 (ii), writing $q = \min\{\frac{1}{2}, \frac{1}{p_2}\}$, keeping in mind our choice of β from (7.22)

$$\begin{aligned}\int_{B_{\theta^{k+1}\rho}(x_0)} |Du|\,dx &\leq \int_{B_{\theta^{k+1}\rho}(x_0)} \left|Du - (Du)_{x_0,\theta^k\rho}\right|dx + \alpha_n\big(\theta^{k+1}\rho\big)\left|(Du)_{x_0,\theta^k\rho}\right| \\ &\leq 2\big(1 + |(Du)_{x_0,\theta^k\rho}|\big)\int_{B_{\theta^k\rho}(x_0)} \frac{|Du - (Du)_{x_0,\theta^k\rho}|}{1 + |\ell_{x_0,\theta^k\rho}|}\,dx \\ &\qquad + \alpha_n\big(\theta^{k+1}\rho\big)^n\left|(Du)_{x_0,\theta^k\rho}\right| \\ &\leq \alpha_n\big(\theta^k\rho\big)^n\left[2\Big(1 + \left|(Du)_{x_0,\theta^k\rho}\right|\Big)⨍_{B_{\theta^k\rho}(x_0)} \frac{|Du - \ell_{x_0,\theta^k\rho}|}{1 + |\ell_{x_0,\theta^k\rho}|}\,dx + \theta^n\left|(Du)_{x_0,\theta^k\rho}\right|\right] \\ &\leq \alpha_n\big(\theta^k\rho\big)^n\left[2\Big(1 + \left|(Du)_{x_0,\theta^k\rho}\right|\Big)\Phi^q(x_0, D\ell_{x_0,\theta\rho}, \rho) + \theta^n\left|(Du)_{x_0,\theta^k\rho}\right|\right] \\ &\leq 2\alpha_n\big(\theta^k\rho\big)^n\Phi^q(x_0, D\ell_{x_0,\theta\rho}, \rho) + \alpha_n\left(\theta^k\rho\right)^n\left[2\Phi^q(x_0, D\ell_{x_0,\theta\rho}, \rho) + \theta^n\right]\left|(Du)_{x_0,\theta^k\rho}\right| \\ &\leq c\big(\theta^k\rho\big)^n\beta + \alpha_n\big(\theta^{k+1}\rho\big)^n ⨍_{B_{\theta^k\rho}(x_0)} |Du|\,dx \\ &\leq c\big(\theta^k\rho\big)^n\beta + \alpha_n\theta^n\int_{B_{\theta^k\rho}(x_0)} |Du|\,dx.\end{aligned}$$

So for each $k \in \mathbb{N}$ and any $\gamma \in (n-1, n)$ there holds

$$\int_{B_{\theta^{k+1}\rho}(x_0)} |Du|\,dx \leq \theta^\gamma\int_{B_{\theta^k\rho}(x_0)} |Du|\,dx + c\beta\big(\theta^k\rho\big)^n,$$

by choice of θ in (7.22). Setting $f(t) = \int_{B_t(x_0)} |Du|\,dx$ in Lemma 3.3, we deduce that for every $r \in (0,\rho)$

$$\begin{aligned}\int_{B_r(x_0)} |Du|\,dx &\leq c\left[r^\gamma + \left(\frac{r}{\rho}\right)^\gamma \int_{B_\rho(x_0)} |Du|\,dx\right] \\ &\leq c\left[1 + \frac{1}{\rho^\gamma}\int_\Omega |Du|\,dx\right] r^\gamma,\end{aligned} \tag{7.28}$$

for some constant depending on θ, γ, n and so ultimately on all of the structure data. Note this estimate holds uniformly at every point $x_0 \in \Omega$ provided (7.16) is in place for a given ρ.

Interpolation

In order to show the conclusion of Theorem 7.1, we require estimates on our renormalised first-order excess functional in terms of the quantities appearing in the characterisation of our singular sets. Similar to the procedure found in Chapter 6, we now use the higher integrability of the solution's gradient and an interpolation argument to achieve this.

We note via Lemma 3.4 (iv) and Hölder's inequality

$$\begin{aligned}\Phi(x_0, D\ell_{x_0,\rho}, \rho) &\leq \fint_{B_\rho(x_0)} \left|\frac{Du - D\ell_{x_0,\rho}}{1 + |D\ell_{x_0,\rho}|}\right|^{p_2} dx + \fint_{B_\rho(x_0)} \left|\frac{Du - D\ell_{x_0,\rho}}{1 + |D\ell_{x_0,\rho}|}\right|^{\min\{p_2,2\}} dx \\ &\leq \fint_{B_\rho(x_0)} \left|\frac{Du - D\ell_{x_0,\rho}}{1 + |D\ell_{x_0,\rho}|}\right|^{p_2} dx + \left(\fint_{B_\rho(x_0)} \left|\frac{Du - D\ell_{x_0,\rho}}{1 + |D\ell_{x_0,\rho}|}\right|^{p_2} dx\right)^{\min\{\frac{2}{p_2},1\}} \\ &\leq 2\left(\fint_{B_\rho(x_0)} \left|\frac{Du - D\ell_{x_0,\rho}}{1 + |D\ell_{x_0,\rho}|}\right|^{p_2} dx\right)^{\min\{\frac{2}{p_2},1\}} \\ &\leq c(p_2)\left(\fint_{B_\rho(x_0)} \left|\frac{Du - D\ell_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|}\right|^{p_2} dx + \left|\frac{(Du)_{x_0,\rho} - D\ell_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|}\right|^{p_2}\right)^{\min\{\frac{2}{p_2},1\}}.\end{aligned} \tag{7.29}$$

Using Lemma 3.9 and Poincaré's inequality we continue to calculate

$$\begin{aligned}\left|\frac{(Du)_{x_0,\rho} - D\ell_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|}\right|^{p_2} &\leq c(n,p_2)\fint_{B_\rho(x_0)} \left|\frac{u - (u)_{x_0,\rho} - (Du)_{x_0,\rho}(x - x_0)}{\rho(1 + |(Du)_{x_0,\rho}|)}\right|^{p_2} dx \\ &\leq c(n,p_2)\fint_{B_\rho(x_0)} \left|\frac{Du - (Du)_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|}\right|^{p_2} dx.\end{aligned} \tag{7.30}$$

We now use the interpolation estimate Lemma 4.4, with $s = p_2$, $p = 1$ and $q = p_2(1 + \frac{\delta}{4})$, we find $\theta = \frac{\delta}{p_2(4+\delta)-4} \in (0,1)$ to compute

$$\left\| \frac{Du - (Du)_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|} \right\|_{L^{p_2}} \leq \left\| \frac{Du - (Du)_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|} \right\|_{L^1}^{\theta} \left\| \frac{Du - (Du)_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|} \right\|_{L^{p_2(1+\frac{\delta}{4})}}^{1-\theta}.$$

After averaging, this is just

$$\begin{aligned}
\fint_{B_\rho(x_0)} \left| \frac{Du - (Du)_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|} \right|^{p_2} dx &\leq \left(\fint_{B_\rho(x_0)} \left| \frac{Du - (Du)_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|} \right| dx \right)^{\frac{p_2\delta}{p_2(4+\delta)-4}} \\
&\quad \times \left(\fint_{B_\rho(x_0)} \left| \frac{Du - (Du)_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|} \right|^{p_2(1+\frac{\delta}{4})} dx \right)^{\frac{4p_2-4}{p_2(4+\delta)-4}} \\
&\leq \kappa^{\frac{p_2\delta}{p_2(4+\delta)-4}} \left(\fint_{B_\rho(x_0)} \left| \frac{Du - (Du)_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|} \right|^{p_2(1+\frac{\delta}{4})} dx \right)^{\frac{4p_2-4}{p_2(4+\delta)-4}},
\end{aligned} \tag{7.31}$$

since $\frac{p_2\delta}{p_2(4+\delta)-4} + \frac{4p_2-4}{p_2(4+\delta)-4} = 1$. Now writing $\lambda = \left(1 + |(Du)_{x_0,\rho}|\right)^{-1}$, we have via Corollary 4.2 (with $p_0 = p_2(1+\frac{\delta}{4})$ and $p = p_2$), and Corollary 4.3 (i) and (ii)

$$\begin{aligned}
\fint_{B_\rho(x_0)} \left| \frac{Du - (Du)_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|} \right|^{p_2(1+\frac{\delta}{4})} dx &\leq c(p_2) \fint_{B_\rho(x_0)} |\lambda Du|^{p_2(1+\frac{\delta}{4})} + \left|\lambda (Du)_{x_0,\rho}\right|^{p_2(1+\frac{\delta}{4})} dx \\
&\leq c(p_2) + c(p_2)\lambda^{p_2(1+\frac{\delta}{4})} \fint_{B_\rho(x_0)} |Du|^{p_2(1+\frac{\delta}{4})} dx \\
&\leq c(p_2) + c\lambda^{p_2(1+\frac{\delta}{4})} \left(\fint_{B_{2\rho}(x_0)} 1 + |Du|^{p(x)} dx \right)^{1+\frac{\delta}{4}} \\
&\leq c(p_2) + c\lambda^{p_2(1+\frac{\delta}{4})} \left(\fint_{B_{2\rho}(x_0)} 1 + |Du|^{p_2} dx \right)^{1+\frac{\delta}{4}} \\
&\leq c(p_2) + c|\lambda|^{p_2(1+\frac{\delta}{4})}(1 + |(Du)_{x_0,2\rho}|)^{p_2(1+\frac{\delta}{4})} \\
&\leq c(p_2) + c|\lambda|^{p_2(1+\frac{\delta}{4})}(1 + |(Du)_{x_0,\rho}|)^{p_2(1+\frac{\delta}{4})} \\
&\leq c =: c_v,
\end{aligned}$$

by definition of λ. Here, the constant retains the dependencies of the constant from Corollary 4.2. Plugging this into (7.31), we have

$$\begin{aligned}
\fint_{B_\rho(x_0)} \left| \frac{Du - (Du)_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|} \right|^{p_2} dx &\leq \left(\fint_{B_\rho(x_0)} \left| \frac{Du - (Du)_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|} \right| dx \right)^{\frac{p_2\delta}{p_2(4+\delta)-4}} \\
&\quad \times \left(\fint_{B_\rho(x_0)} \left| \frac{Du - (Du)_{x_0,\rho}}{1 + |(Du)_{x_0,\rho}|} \right|^{p_2(1+\frac{\delta}{4})} dx \right)^{\frac{4p_2-4}{p_2(4+\delta)-4}} \\
&\leq \kappa^{\frac{p_2\delta}{p_2(4+\delta)-4}} c_v^{\frac{4p_2-4}{p_2(4+\delta)-4}}.
\end{aligned} \tag{7.32}$$

We note the dependence of κ on γ (via β's dependence on θ), and note that $\kappa \to 0$ as $\gamma \to n$. Furthermore, note that the condition

$$\rho \fint_{B_\rho(x_0)} |Du|\, dx < \sigma,$$

together with Poincaré's inequality, implies

$$\begin{aligned}
\Upsilon(x_0, \ell_{x_0,\rho}, \rho) &= \omega_\xi \left(\fint_{B_\rho(x_0)} |u - (u)_{x_0,\rho}|\, dx \right) \\
&\leq \omega_\xi \left(\rho \fint_{B_\rho(x_0)} |Du|\, dx \right) \\
&\leq \omega_\xi\,(\sigma) \\
&\leq \iota.
\end{aligned}$$

Taking some point $x_0 \in \Omega$ and $\rho \leq \hat{\rho}$ satisfying $|D\ell| \leq M$ for fixed $M < \infty$

$$\liminf_{\rho \downarrow 0} \fint_{B_\rho(x_0)} \left| \frac{Du - (Du)_{x_0,\rho}}{1 + |\,(Du)_{x_0,\rho}\,|} \right| dx < \kappa, \qquad \text{and} \qquad \liminf_{\rho \downarrow 0} \rho \fint_{B_\rho(x_0)} |Du|\, dx < \sigma, \quad (7.33)$$

we can find a $\rho < \hat{\rho}$ such that the conditions of Lemma 7.7 hold, with ρ at this stage depending on all of the structure conditions. Furthermore, if the conditions of Lemma 7.7 hold at this point x_0 and fixed $\rho < \hat{\rho}$, then by the absolute continuity of the Lebesgue integral, there exists an $R < \rho$ such that these same conditions hold for each $x \in B_R(x_0)$. Consequently, we deduce that (7.28) and (7.32) hold for every $r \leq \frac{R}{4}$ and $y \in B_{\frac{R}{4}}(x_0)$. This implies Du belongs to the Morrey space $L^{1,\gamma}(B_{\frac{R}{4}}(y), \mathbb{R}^{nN})$ for $\gamma \in (n-1, n)$, and so the Morrey-Campanato embedding theorem implies $u \in C^{0,\tau}(B_{\frac{R}{4}}(y), \mathbb{R}^N)$ for $\tau = 1 - n - \gamma$. Note τ can be chosen to be any value in $(0,1)$ provided κ and hence σ are chosen accordingly as functions of τ, decaying to 0 as $\alpha \uparrow 1$. This in turn restricts the neighbourhood $B_{\frac{R}{4}}(x_0)$ on which the estimate holds, since $\hat{\rho}$ also has dependence on γ via α through β, and so ultimately θ. Indeed, if we take $\kappa = \sigma = 0$ we can obtain the conclusion of Theorem 7.1 for every $\tau \in (0,1)$.

By definition we have the inclusions $\Sigma^\tau_{1,u} \subset \Sigma^0_{1,u}$ and $\Sigma^\tau_{2,u} \subset \Sigma^0_{2,u}$. Since these sets are closed, we have $\operatorname{Reg} u$ and $\operatorname{Reg}^\tau(u)$ are relatively open in Ω and hence open. Furthermore, by Lebesgue's differentiation theorem we have $|\Sigma^0_{1,u}| = 0$, and Lemma 3.1 implies $\dim_{\mathcal{H}}(\Sigma^0_{2,u}) \leq n - 1$. We conclude that $|\Omega \setminus \operatorname{Reg}(u)| = 0$ and hence $|\Omega \setminus \operatorname{Reg}^\tau(u)| = 0$. ■

Chapter 8

Systems with continuous coefficients in low dimensions

In this chapter, we present proof of Theorem 2.4. We show improved partial Hölder continuity of weak solutions to the inhomogeneous systems of nonlinear elliptic PDE in divergence form

$$\begin{cases} -\operatorname{div} a(x,u,Du) = b(x,u,Du) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{8.1}$$

for some given boundary data function $g \in C^{1,\alpha}(\overline{\Omega},\mathbb{R}^N)$, under suitable restrictions on the domain. As usual, a weak solution is any function $u \in W^{1,p(\,\cdot\,)}(\Omega,\mathbb{R}^N)$ satisfying $u\,|_{\partial\Omega}= g$ in the trace sense (see §12.1 in [DHHR11]), and

$$\int_\Omega a(x,u,Du)D\phi\,dx = \int_\Omega b(x,u,Du)\phi\,dx$$

for any fixed $\phi \in W_0^{1,p(\,\cdot\,)}(\Omega,\mathbb{R}^N)$.

Similar to Chapter 6, we will consider regularity up to the boundary, and so in practice deal with the transformed operator on the model problem

$$\begin{cases} -\operatorname{div} \hat{a}(x,u,Du) = \hat{b}(x,u,Du) & \text{in } B^+, \\ u = g & \text{on } \Gamma. \end{cases} \tag{8.2}$$

Note that unlike the situation in Chapter 6, the boundary data in (8.2) is not in general vanishing. This greatly simplifies the calculations in Chapter 5 and so these calculations will not be repeated.

Our weak solution is of course now interpreted as a function $u \in W^{1,p(\,\cdot\,)}(B^+,\mathbb{R}^N)$ that satisfies

$$\int_{B^+} a(x,u,Du)D\phi\,dx = \int_{B^+} b(x,u,Du)\phi\,dx$$

for any given test function $\phi \in W_0^{1,p(\,\cdot\,)}(B^+,\mathbb{R}^N)$. As in Chapter 6, the operators and inhomogeneities a and $\hat{a}$, and b and $\hat{b}$ differ only in structure parameters, and for ease of notation we will consider them to be identical.

We will assume the log-Hölder continuity condition on the exponent $p : \Omega \to [\gamma_1, \gamma_2]$. That is, there exists an $L > 0$ such that

$$\limsup_{\rho \downarrow 0} \omega_p(\rho) \log \left(\frac{1}{\rho} \right) \leq L.$$

We note that when dealing with Euler-Lagrange systems we must strengthen this assumption to vanishing log-Hölder continuity. That is,

$$\lim_{\rho \downarrow 0} \omega_p(\rho) \log \left(\frac{1}{\rho} \right) = 0. \tag{8.3}$$

This is a stronger condition than is strictly needed - we require only the existence of some small $\delta_p > 0$ such that

$$\limsup_{\rho \downarrow 0} \omega_p(\rho) \log \left(\frac{1}{\rho} \right) \leq \delta_p. \tag{8.4}$$

We also assume the natural energy bound on the solution

$$\int_\Omega |Du|^{p(x)} \leq E < \infty. \tag{8.5}$$

The operator $a : \Omega \times \mathbb{R}^N \times \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$ is a Carathédory vector field, satisfies the following. For fixed $0 < \nu \leq L < \infty$, all triples $(x, \xi, z) \in \Omega \times \mathbb{R}^N \times \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$, and any $\zeta \in \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$:

(L1) Strong uniform ellipticity: $\nu(1 + |z|)^{p(x)-2}|\zeta|^2 \leq D_z a(x, \xi, z)\zeta \cdot \zeta$,

(L2) Nonstandard $p(x)$ growth: $|a(x, \xi, z)| \leq L(1 + |z|)^{p(x)-1}$,

(L3) Bounded derivatives in z: $|D_z a(x, \xi, z)| \leq L(1 + |z|)^{p(x)-2}$,

(L4) Hölder continuity in u: $|a(x, \xi, z) - D_z a(x, \hat{\xi}, z)| \leq L\omega_\xi(|\xi - \hat{\xi}|)(1 + |z|)^{p(x)-1}$,

(L5) Continuity in x:

$$\begin{aligned} |a(x, \xi, z) - a(y, \xi, z)| \leq L\omega\left(|x - y|\right) &\left[(1 + |z|)^{p(x)-1} + (1 + |z|)^{p(y)-1}\right] \\ &\times \left[1 + \log\left(1 + |z|\right)\right]. \end{aligned}$$

Here we assume both $\omega, \omega_\xi : [0, \infty) \to [0, 1]$, where ω_ξ additionally satisfies $\omega_\xi(t) \leq \min\{t^\alpha, 1\}$ for some fixed $\alpha \in (0, 1)$. We assume that ω satisfies the vanishing log-Hölder continuity assumption (8.3).

We take the inhomogeneity b to obey either

(G1) Controllable growth: $b(x,\xi,z) \leq L_0(1+|z|)^{p(x)-1}$.

or

(G2) Natural growth for bounded solutions: $b(x,\xi,z) \leq L_1|z|^{p(x)} + L_2$,

for $L_1, L_2 > 0$ where $L_1 = L_1(\|u\|_{L^\infty})$ satisfies $2L_1\|u\|_{L^\infty} < \nu$.

We note that **(L1)** implies the coercivity condition

(C) $\nu(1+|z|^2)^{p(x)} - C \leq a(x,\xi,z)\cdot z$.

Statement of main result

Theorem 8.1. *Fix $g \in C^1$ and let $u \in g + W_\Gamma^{1,p(\,\cdot\,)}(\Omega,\mathbb{R}^N)$ be a weak solution to (8.1) under assumptions* **(L1)**–**(L5)**, *with the inhomogeneity satisfying either the controllable growth condition* **(G1)**, *or the natural growth condition* **(G2)** *with the additional assumption that the solution satisfies the bound $2L_1\|u\|_{L^\infty} < \nu$. Then there exists an $\varepsilon_0 > 0$ such that on the subset Ω_p of Ω where $n-2-\frac{\varepsilon_0}{2} < p(x)$ there holds*

$$\dim_{\mathcal{H}}\big(\operatorname{Sing}_u(\Omega_p)\big) < n - \gamma_1.$$

Moreover, we have

$$u \in C^{0,\gamma}_{loc}\big(\operatorname{Reg}_u(\Omega_p)\big),$$

for all $\gamma \in \left(0, \min\left\{1 - \frac{n-2-\frac{\varepsilon_0}{2}}{\gamma_1}, 1\right\}\right)$. Furthermore, we can characterise the singular set via the enclosure

$$\operatorname{Sing}_u(\Omega_p) \subset \Sigma_{p,\Omega_p} := \left\{x \in \overline{\Omega} : \liminf_{\rho\downarrow 0} \rho^{p(x)-n}\mathcal{M}_p(x,r) > 0\right\}.$$

Here, $\gamma_1 = \inf_{x\in\Omega} p(x)$. Note that for $p(x)(1+\frac{\delta}{2}) > n$ we combine the higher integrability result with the Sobolev-Morrey embedding theorem to attain everywhere Hölder continuity. On the other hand, when $p(x) < n-2-\frac{\varepsilon_0}{2}$ we have almost everywhere Hölder continuity via Theorem 2.3.

Remark 8.2. *In fact, we can provide a local improvement of this characterisation, since by careful tracking through the calculations it is evident that for any point $x \in \operatorname{Reg}_u(\Omega_p)$ we have*

$$u \in C^{0,\hat{\gamma}}\big(N\big),$$

for all $\gamma \in \left(0, \min\left\{1 - \frac{n-2-\frac{\varepsilon_0}{2}}{p_M}, 1\right\}\right)$ and some open neighbourhood N of x. Here, of course $p_M = p_M(x) = \sup_{B_{\rho_0}(x)\cap\overline{\Omega}} p(y)$, where ρ_0 is the radius given in Corollary 4.2.

After introducing some preliminary estimates, we will consider the two growth assumptions on the inhomogeneities separately, as the natural growth assumption will require testing with a different class of function, first developed by Arkhipova in [Ark03].

Comparison map

In order to obtain decay estimates, we show the solution u lies 'close' to the solution to a related, frozen coefficient PDE.

We define now the operator $a_o : \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N) \to \mathrm{Hom}(\mathbb{R}^n \otimes \mathbb{R}^N)$, corresponding to a frozen at the point $\hat{x}$, satisfying

$$a_o(z) = a(\hat{x}, (u)^+_{x_0,\rho}, z).$$

As usual we have taken $\hat{x}$ satisfying $p(\hat{x}) = \sup_{x \in \overline{B^+_\rho(x_0)}} p(x)$. We will consider weak solutions $v \in W^{1,p_2}_\Gamma(B^+_\rho(x_0), \mathbb{R}^N)$ to the frozen system

$$\begin{cases} \operatorname{div} a_o(Dv) = 0 & \text{in } \; B^+_\rho(0), \\ \qquad\qquad v = u - g & \text{on } \; \partial B^+_\rho(0), \end{cases} \tag{8.6}$$

From Corollary 4.6 in [Bec08] we have the following $p-$Dirichlet energy estimate, noting that the proof in the superquadratic case is identical if we use Theorem 3.1 from [Cam87a] in place of Lemma 4.5 in [Bec08]. It should be noted that we take $t_0(n, N, \frac{L}{\nu}, p_2) > 1$ to be the higher integrability exponent from §4.3.2 in [Bec08]. The dependence on p_2 is continuous, so we may assume $t_0 = t_0(n, N, \frac{L}{\nu}, \gamma_1, \gamma_2)$. Note that its derivation does not require that $1 < p < 2$, and holds for all $1 < p < \infty$.

Decay estimates

Lemma 8.3. *Let $v \in W^{1,p_2}_\Gamma(B^+_\rho(x_0), \mathbb{R}^N)$ be a weak solution to* (8.6) *under assumptions* **(L1)–(L5)**. *Then there exists a constant c depending only on $(n, N, \gamma_1, \gamma_2, L, \nu)$, such that for every $B^+_r(y) \subset B^+_\rho(x_0)$ with centre $y \in B^+_\rho(x_0) \cup \Gamma_\rho(x_0)$ and radius $0 < r < \rho - |x_0 - y|$, for all $0 < \tau \le 1$ there holds*

$$\int_{B^+_{\tau r}(y)} 1 + |Dv|^{p_2}\, dx \le c\tau^\mu \int_{B^+_r(y)} 1 + |Dv|^{p_2}\, dx.$$

Here, $\mu := \min\{n, 2 + \varepsilon_0\}$ for $\varepsilon_0 = n(1 - \frac{1}{t_0}) > 0$. Further,

$$\int_{B^+_r(x_0)} 1 + |Dv|^{p_2}\, dx \le c\left(\frac{r}{\rho}\right)^\mu \int_{B^+_\rho(x_0)} 1 + |Dv|^{p_2}\, dx.$$

From Corollary 4.7 in [Bec08] for the subquadratic case, and Theorem 1.II in [Cam87a] for the superquadratic case, we have the following

Corollary 8.4. *Let $v \in W^{1,p_2}_\Gamma(B^+_\rho(x_0), \mathbb{R}^N)$ be a weak solution to* (8.6) *under assumptions* **(L1)–(L5)**, *with $n \in [2, p_2 + \mu)$. Then for every $B^+_{\tau r}(y) \subset B^+_\rho(x_0)$ with centre $y \in B^+_\rho(x_0) \cup \Gamma_\rho(x_0)$ and radius $0 < r < \rho - |x_0 - y|$, and for all $0 < \tau < 1$ there holds*

$$\int_{B^+_{\tau r}(y)} |v|^{p_2}\, dx \leq c\tau^n \left[\int_{B^+_r(y)} |v|^{p_2}\, dx + \rho^{p_2} \int_{B^+_r(y)} 1 + |Dv|^{p_2}\, dx \right].$$

Here the constant c depends only on n, N, p_2 and $\frac{L}{\nu}$.

Comparison estimate

The following estimate lets us to compare the p-Dirichlet energy of the comparison map v with that of our solution u. The fixed exponent analogues appear in §3 of [Bec08] for the subquadratic case, and the superquadratic case can be deduced from the proof of Theorem 1 in [Ark03].

Lemma 8.5. *Let $v \in W^{1,p_2}_\Gamma(B^+_\rho(x_0), \mathbb{R}^N)$ be a weak solution to* (8.6) *under assumptions* **(L1)–(L5)**. *Then there exists a constant c depending only on $\gamma_1, \gamma_2, \frac{L}{\nu}, ||Dg||_{L^\infty}$, such that there holds*

$$\int_{B^+_\rho(x_0)} |Dv|^{p_2}\, dx \leq c\Big(p_2, \frac{L}{\nu}, \|Dg\|_{L^\infty}\Big) \int_{B^+_\rho(x_0)} 1 + |Du|^{p_2}\, dx.$$

Proof of Lemma 8.5: Following [Ark03], we test (8.6) with $v - (u - g)$ and use **(C)** and **(L1)** on the left, then **(L2)** and Young's inequality $(p_2, \frac{p_2}{p_2 - 1})$ to calculate

$$\begin{aligned}
\nu \int_{B^+_\rho(x_0)} |Dv|^{p_2} - C\, dx &\leq \nu \int_{B^+_\rho(x_0)} \big(1 + |Dv|^2\big)^{\frac{p_2}{2}}\, dx - C \\
&\leq \int_{B^+_\rho(x_0)} a_0(Dv) \cdot Dv\, dx \\
&= \int_{B^+_\rho(x_0)} a_0(Dv) \cdot (Du - Dg)\, dx \\
&\leq L \int_{B^+_\rho(x_0)} \big(1 + |Dv|^2\big)^{\frac{p_2 - 1}{2}} |Du - Dg|\, dx \\
&\leq \varepsilon \int_{B^+_\rho(x_0)} 1 + |Dv|^{p_2}\, dx + c(p_2, \varepsilon, L) \int_{B^+_\rho(x_0)} |Du - Dg|^{p_2}\, dx \\
&\leq \varepsilon \int_{B^+_\rho(x_0)} 1 + |Dv|^{p_2}\, dx + c(p_2, \varepsilon, L, \|Dg\|_{L^\infty}) \int_{B^+_\rho(x_0)} 1 + |Du|^{p_2}\, dx.
\end{aligned}$$

Fixing $\varepsilon = \frac{\nu}{2}$, then absorbing the common terms and renormalising, we have the result. □

Morrey space regularity

The goal of this section is to estimate the Morrey space norm of the gradient Du of the bounded solution to (8.1) under condition **(G2)**. We begin with a Caccioppoli type inequality, which is not required to be as sharp as the ones derived in the previous sections. We note that this Caccioppoli inequality is of the type used in the proof of Lemma 4.1.

A Caccioppoli Inequality

Lemma 8.6. *Let $u \in g + W_\Gamma^{1,p(\cdot)} \cap L^\infty(B_\rho^+(x_0), \mathbb{R}^N)$ be a weak solution to (8.1) under assumptions* **(L1)**–**(L3)**, *with the inhomogeneity satisfying the natural growth condition* **(G2)**, *and $g \in C^1$.*

Then we have the estimate

$$\int_{B_{\frac{\rho}{2}}(x_0)} 1 + |Du|^{p(x)}\, dx \leq C \int_{B_\rho(x_0)} 1 + \left| \frac{u - (u)^+_{x_0,\rho}}{\rho} \right|^{p(x)} dx.$$

for any $\rho \leq \tilde{\rho}$, where C and $\tilde{\rho}$ depend on $\gamma_1, \gamma_2, \nu, L, L_1, L_2, \|u\|_{L^\infty}$ and $\|g\|_{C^1}$.

Proof of Lemma 8.6: Taking a standard cutoff function $\eta \in C_0^\infty(B_\rho(x_0))$ satisfying $0 \leq \eta \leq 1$, $\eta = 1$ on $B_{\frac{\rho}{2}}(x_0)$, $\eta = 0$ outside $B_{\frac{3\rho}{4}}(x_0)$ and $|D\eta| \leq \frac{C}{\rho}$, we write $\phi := \eta^{\hat{p}}(u - g)$ where $\hat{p} = \max\{2, p_2\}$. Note that $u = g$ on $\Gamma_\rho(x_0)$ and so $\phi \in W_\Gamma^{1,p(\cdot)}(B^+_{\rho_0}(0), \mathbb{R}^N)$ is an admissible test function, with

$$D\phi = \hat{p}\eta^{\hat{p}-1}(u - g) \otimes D\eta + \eta^{\hat{p}}(Du - Dg). \tag{8.7}$$

Testing (8.1) with ϕ we find

$$\begin{aligned}
\fint_{B_\rho^+(x_0)} a(x, u, Du) \cdot \eta^{\hat{p}} Du\, dx \ &= \fint_{B_\rho^+(x_0)} a(x, u, Du) \cdot \left(\hat{p}\eta^{\hat{p}-1}(u - g) \otimes D\eta + \eta^{\hat{p}} Dg\right) dx \\
&\qquad + \fint_{B_\rho^+(x_0)} b(x, u, Du)\eta^{\hat{p}}(u - g)\, dx \\
&\leq \fint_{B_\rho^+(x_0)} \left|a(x, u, Du)\right| \cdot \left|\hat{p}\eta^{\hat{p}-1}(u - g) \otimes D\eta + \eta^{\hat{p}} Dg\right| dx \\
&\qquad + \fint_{B_\rho^+(x_0)} \left|b(x, u, Du)\right| \eta^{\hat{p}} \left|u - g\right| dx,
\end{aligned}$$

with the obvious notation

$$\mathrm{I} \leq \mathrm{II} + \mathrm{III}.$$

Now, **(C)** immediately implies

$$\mathrm{I} = \fint_{B_\rho^+(x_0)} a(x,u,Du)\cdot \eta^{\hat{p}} Du\,dx \ \geq \nu \fint_{B_\rho^+(x_0)} \eta^{\hat{p}}(1+|Du|)^{p(x)} - C\,dx.$$

For II, we can use **(L2)**, then Young's inequality with pointwise exponents $(p(x), \frac{p(x)}{p(x)-1})$ to find

$$\begin{aligned}
\mathrm{II} &= \fint_{B_\rho^+(x_0)} \big|a(x,u,Du)\big|\cdot\big|\hat{p}\eta^{\hat{p}-1}(u-g)\otimes D\eta + \eta^{\hat{p}} Dg\big|\,dx \\
&\leq c\fint_{B_\rho^+(x_0)} (1+|Du|)^{p(x)-1}\eta^{\hat{p}-1}\left[\left|\frac{u-g}{\rho}\right| + |Dg|\right] dx \\
&\leq \varepsilon \fint_{B_\rho^+(x_0)} (1+|Du|)^{p(x)}\eta^{\hat{p}}\,dx + C(\varepsilon,\gamma_1,\gamma_2)\fint_{B_\rho^+(x_0)} \left|\frac{u-g}{\rho}\right|^{p(x)} dx + c(\|Dg\|_{L^\infty},\gamma_1,\gamma_2).
\end{aligned}$$

This leaves us with term III, and without losing generality we restrict our domain such that

$$\rho < \tilde{\rho} := \min\left\{\rho_0 - |x_0|, \frac{\nu - 2\|u\|_{L^\infty}}{8L_1\|Dg\|_{L^\infty}+1}\right\}.$$

Write $x' = (x_1, ..., x_{n-1}, 0)$ as the projection of $x \in \mathbb{R}^n$ onto $\mathbb{R}^{n-1}\times\{0\}$. We now calculate, by **(G2)** with $2L_1\|u\|_{L^\infty} < \nu$, keeping in mind the differentiability of g,

$$\begin{aligned}
\mathrm{III} &= \fint_{B_\rho^+(x_0)} \big|b(x,u,Du)\big|\eta^{\hat{p}}\big|u-g\big|\,dx \\
&\leq \fint_{B_\rho^+(x_0)} L_1(1+|Du|)^{p(x)}\eta^{\hat{p}}\big(\big|u(x)-g(x')\big| + \big|g(x')-g(x)\big|\big) + L_2|u-g|\,dx \\
&\leq 2L_1\big(\|u\|_{L^\infty} + \rho\|Dg\|_{L^\infty}\big)\fint_{B_\rho^+(x_0)} (1+|Du|)^{p(x)}\eta^{\hat{p}}\,dx + c(L_2,\|u\|_{L^\infty},\|g\|_{L^\infty}).
\end{aligned}$$

Combining these terms, we have

$$\begin{aligned}
\nu\fint_{B_\rho^+(x_0)} \eta^{\hat{p}}(1+|Du|)^{p(x)}\,dx \leq C\fint_{B_\rho^+(x_0)} 1 + \left|\frac{u-g}{\rho}\right|^{p(x)} dx& \\
+ 2L_1\big(\|u\|_{L^\infty} + \rho\|Dg\|_{L^\infty} + \varepsilon\big)\fint_{B_\rho^+(x_0)} (1+|Du|)^{p(x)}\eta^{\hat{p}}\,dx&,
\end{aligned}$$

and so fixing $\varepsilon < \frac{\nu - 2\|u\|_{L^\infty}}{4L_1\|Dg\|_{L^\infty}+1}$ we conclude

$$\fint_{B_\rho^+(x_0)} \eta^{\hat{p}}(1+|Du|)^{p(x)}\,dx \leq C\fint_{B_\rho^+(x_0)} 1 + \left|\frac{u-g}{\rho}\right|^{p(x)} dx,$$

where the constant depends only on $\gamma_1, \gamma_2, \nu, L, L_1, L_2, \|u\|_{L^\infty}$ and $\|g\|_{C^1}$. $\square$

A Morrey space estimate

Lemma 8.7. *Let $u \in g + W^{1,p(\cdot)}_\Gamma(B^+_\rho(x_0), \mathbb{R}^N) \cap L^\infty(B^+_\rho(x_0), \mathbb{R}^N)$ be a weak solution to* (8.1) *with* (8.4) *and* (8.5) *under assumptions* **(L1)**–**(L3)**, *with the inhomogeneity satisfying the natural growth condition* **(G2)**, *with $2L_1\|u\|_{L^\infty} < \nu$ and $g \in C^1$. Furthermore, restrict $2\tau\rho < \tilde\rho < 1$, where $\tilde\rho$ is the constant from Lemma 8.6. Then $Du \in L^{p_2, n-p_2}(B^+_{(1-2\tau)\rho/2}, \mathbb{R}^{nN})$ and*

$$\|Du\|^{p_2}_{L^{p_2,n-p_2}(B^+_{(1-2\tau)\rho}, \mathbb{R}^{nN})} \leq C\Big(1 + \|u\|^{p_2}_{L^\infty(B^+_\rho, \mathbb{R}^N)}\Big).$$

for any $\rho \leq \tilde\rho$, where the constant C depends on $n, N, L/\nu, \gamma_1, \gamma_2, L_1, L_2, E,$ and ω_p.

Proof of Lemma 8.7: We use Corollary 4.2, Lemma 8.6 and the assumption $u \in L^\infty$ to compute

$$\begin{aligned}
\|Du\|^{p_2}_{L^{p_2,n-p_2}} &= \sup_{x \in B^+_{(1-2\tau)\rho/2}(x_0), 0<r<\rho} r^{p_2-n} \int_{B_r(x) \cap B^+_{(1-2\tau)\rho/2}} |Du|^{p_2}\, dy \\
&\leq c \sup_{x \in B^+_{(1-2\tau)\rho}(x_0), 0<r<\rho} r^{p_2} \fint_{B_r(x) \cap B^+_{\rho/2}(x_0)} 1 + |Du|^{p(x)}\, dy \\
&\leq c \sup_{x \in B^+_{(1-2\tau)\rho}(x_0), 0<r<\rho} r^{p_2} \fint_{B_r(x) \cap B^+_{\rho}(x_0)} 1 + \left| \frac{u - (u)^+_{x_0,\rho}}{r} \right|^{p(x)} dx \\
&\leq c \sup_{x \in B^+_{(1-2\tau)\rho}(x_0), 0<r<\rho} r^{p_2} \fint_{B_r(x) \cap B^+_{\rho}(x_0)} 1 + \left| \frac{u - (u)_{B_r(x) \cap B^+_\rho(x_0)}}{r} \right|^{p_2} dx \\
&\leq c\big(1 + \|u\|^{p_2}_{L^\infty}\big).
\end{aligned}$$

□

A Morrey-type functional

Recall from (3.14) the Morrey-type functional $\mathcal{M} : B_\rho(x_0) \times (0, R]$, satisfying

$$\mathcal{M}(x, r) := \int_{B_r(x)} 1 + |Du|^{p_2}\, dy, \tag{8.8}$$

and

$$\mathcal{M}_p(x, r) := \int_{B_r(x)} 1 + |Du|^{p(y)}\, dy, \tag{8.9}$$

and its analogous boundary version. We treat both these versions as the same object, since the domain of definition is clear from context.

Controllable growth

We will treat systems with inhomogeneities obeying **(G1)** and **(G2)** separately. First consider the simpler case where the inhomogeneity satisfies **(G1)**. For $x \in \mathbb{R}^n$ we write $x = (x_1, ..., x_n)$. In this case, we have the following

Lemma 8.8. *Let $u \in g + W^{1,p(\,\cdot\,)}_{\Gamma}(B^+, \mathbb{R}^N)$ be a weak solution to* (8.1) *with* (8.4) *and* (8.5) *under assumptions* **(L1)**–**(L3)**, *with the inhomogeneity satisfying the controllable growth condition* **(G1)**, *for $g \in C^1$. Fix $\sigma = \frac{(p_2-1)\delta_0}{p_2(4+\delta_0)}$ and $\mu := \min\{n, 2+\varepsilon_0\}$ for $\varepsilon_0 = n(1-\frac{1}{s}) > 0$, where s is the higher integrability exponent from Corollary 4.2. For $x_0 \in \Gamma, \rho < 1-|x_0|$ or $x_0 \in B^+, \rho < \min\{1-|x_0|, (x_0)_n\}$, and $r < \rho$ we have*

$$\mathcal{M}(x_0, r) \le c\left[\left(\frac{r}{\rho}\right)^{\mu} + \hat{\omega}^{\sigma}\Big(\hat{c}\big(\rho^{p_2-n}\mathcal{M}(x_0,\rho)\big)^{\frac{1}{p_2}}\Big) + \rho + \delta + \delta_p\right]\mathcal{M}(x_0,\rho) + c\delta^{1-p_2}\rho^n, \tag{8.10}$$

for each $0 < r < \rho$. The constant c depends on $n, N, L/\nu, \gamma_1, \gamma_2, L_0, E, \omega_p\ \|Dg\|_\infty$, while $\hat{c}$ depends only on n and p_2.

Proof of Lemma 8.8: As usual, we prove only the boundary case, since the interior estimates are analogous and in fact simpler. Recalling Lemma 3.6, elementary integration with **(L1)**, and noting $u - g - v$ vanishes on the boundary, where v solves (8.6), we can estimate

$$\begin{aligned}
&\nu \int_{B_\rho^+(x_0)} (1 + |Du| + |Dv + Dg|)^{p_2-2}|Du - Dv - Dg|^2\,dx \\
&\le c\int_{B_\rho^+(x_0)} \int_0^1 (1 + |Du + t(Dv + Dg - Du)|)^{p_2-2}|Du - Dv - Dg|^2\,dt\,dx \\
&\le c\int_{B_\rho^+(x_0)} \int_0^1 D_z a_0\big(Du + t(Dv + Dg - Du)\big)(Dv + Dg - Du)\cdot(Dv + Dg - Du)\,dt\,dx \\
&= c\int_{B_\rho^+(x_0)} \big[a_0(Dv + Dg) - a_0(Du)\big]\cdot(Dv + Dg - Du)\,dx \\
&= c\int_{B_\rho^+(x_0)} \big[a_0(Dv + Dg) - a_0(Dv)\big]\cdot(Dv + Dg - Du)\,dx \\
&\qquad + c\int_{B_\rho^+(x_0)} \big[a(x, u, Du) - a(\hat{x}, u, Du)\big]\cdot(Dv + Dg - Du)\,dx \\
&\qquad + c\int_{B_\rho^+(x_0)} \big[a(\hat{x}, u, Du) - a\big(\hat{x}, (u)^+_{x_0,\rho}, Du\big)\big]\cdot(Dv + Dg - Du)\,dx \\
&\qquad + c\int_{B_\rho^+(x_0)} b(x, u, Du)\cdot(u - v - g)\,dx \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},
\end{aligned}$$

with the obvious labelling. Estimating each term separately, we first assume $1 < p < 2$

and use elementary integration with **(L1)**, Lemma 3.6, Young's inequality and Lemma 8.5 to compute

$$
\begin{aligned}
\mathrm{I} &= c\int_{B_\rho^+(x_0)} \big[a_0(Dv+Dg)-a_0(Dv)\big]\cdot(Dv+Dg-Du)\,dx \\
&= c\int_{B_\rho^+(x_0)}\int_0^1 D_z a_0(Dv+tDg)(Dg)\cdot(Dv+Dg-Du)\,dt\,dx \\
&\le c\int_{B_\rho^+(x_0)}\int_0^1 (1+|Dv+tDg|)^{p_2-2}|Dg||Dv+Dg-Du|\,dt\,dx \\
&\le c\int_{B_\rho^+(x_0)} (1+|Dv|+|Dg|)^{p_2-2}|Dg||Dv+Dg-Du|\,dx \\
&\le c\int_{B_\rho^+(x_0)} |Dv+Dg-Du|\,dx \\
&\le c\int_{B_\rho^+(x_0)} 1+|Dv|+|Du|\,dx \\
&\le c\delta\int_{B_\rho^+(x_0)} 1+|Dv|^{p_2}+|Du|^{p_2}+\delta^{1-p_2}\,dx \\
&\le c\delta\int_{B_\rho^+(x_0)} 1+|Du|^{p_2}\,dx + c\rho^n\delta^{1-p_2}
\end{aligned}
$$

for $\delta\in(0,1)$ to be fixed later. On the other hand, when $p\ge 2$ we estimate again using elementary integration the ellipticity condition **(L1)** and Lemma 3.6, then Young's inequality and Lemma 8.5

$$
\begin{aligned}
\mathrm{I} &= c\int_{B_\rho^+(x_0)} \big[a_0(Dv+Dg)-a_0(Dv)\big]\cdot(Dv+Dg-Du)\,dx \\
&= c\int_{B_\rho^+(x_0)}\int_0^1 D_z a_0(Dv+tDg)(Dg)\cdot(Dv+Dg-Du)\,dt\,dx \\
&\le c\int_{B_\rho^+(x_0)}\int_0^1 (1+|Dv+tDg|)^{p_2-2}|Dg||Dv+Dg-Du|\,dt\,dx \\
&\le c\int_{B_\rho^+(x_0)} (1+|Dv|+|Dg|)^{p_2-2}|Dg||Dv+Dg-Du|\,dx \\
&\le c\int_{B_\rho^+(x_0)} (1+|Dv|)^{p_2-2}\big(|Dv+Dg|+|Du|\big)\,dx \\
&\le c\int_{B_\rho^+(x_0)} (1+|Dv|)^{p_2-1}\,dx + c\int_{B_\rho^+(x_0)}(1+|Dv|)^{p_2-2}|Du|\,dx \\
&\le c\int_{B_\rho^+(x_0)} \delta(1+|Dv|^{p_2})+\delta^{1-p_2}\,dx + c\int_{B_\rho^+(x_0)}\delta(1+|Dv|)^{p_2-2\frac{p_2}{p_2-1}}|Du|^{\frac{p_2}{p_2-1}}+\delta^{1-p_2}\,dx \\
&\le c\delta\int_{B_\rho^+(x_0)} 1+|Dv|^{p_2}\,dx + c\delta\int_{B_\rho^+(x_0)}(1+|Dv|)^{(p_2-2)\frac{p_2}{p_2-1}\frac{p_2-1}{p_2-2}}+|Du|^{\frac{p_2}{p_2-1}(p_2-1)}\,dx + c\rho^n\delta^{1-p_2} \\
&\le c\delta\int_{B_\rho^+(x_0)} 1+|Du|^{p_2}\,dx + c\delta^{1-p_2}\rho^n.
\end{aligned}
$$

In estimating II, we use **(L5)**, and Hölder's inequality to find we take our higher integrability exponent δ_0, and noting $\frac{(p_2-1)\delta_0}{p_2(4+\delta_0)}+\frac{4p_2-4}{p_2(4+\delta_0)}+\frac{1}{p_2}=1$, we set $\sigma:=\frac{(p_2-1)\delta_0}{p_2(4+\delta_0)}$. We use **(L5)**, and Hölder's inequality to find

$$
\begin{aligned}
\mathrm{II} &= c\int_{B_\rho^+(x_0)} \big[a(x,u,Du)-a(\hat{x},u,Du)\big]\cdot(Dv+Dg-Du)\,dx \\
&\le c\int_{B_\rho^+(x_0)} \big|a(x,u,Du)-a(\hat{x},u,Du)\big||Dv+Dg-Du|\,dx \\
&\le c\int_{B_\rho^+(x_0)} \omega(x-\hat{x})(1+|Du|)^{p_2-1}\big[1+\log(1+|Du|^2)\big]|Dv+Dg-Du|\,dx \\
&\le c|B_\rho^+|\fint_{B_\rho^+(x_0)} \omega(2\rho)(1+|Du|)^{p_2-1}\left[1+\log\left(1+|Du|^2\right)\right]|Dv+Dg-Du|\,dx \\
&\le c|B_\rho^+|\left(\fint_{B_\rho^+(x_0)} \omega^{\frac{p_2}{p_2-1}}(2\rho)(1+|Du|)^{p_2}\big[1+\log^{\frac{p_2}{p_2-1}}(1+|Du|^2)\big]\,dx\right)^{\frac{p_2-1}{p_2}} \\
&\qquad\qquad\times\left(\fint_{B_\rho^+(x_0)}|Dv+Dg-Du|^{p_2}\,dx\right)^{\frac{1}{p_2}}
\end{aligned}
$$

We continue using Lemma 4.5 (with $\gamma=\frac{p_2}{p_2-1}$), the concavity of log, Lemma 8.5, and the log-Hölder condition (8.4) to compute

$$\begin{aligned}
\mathrm{II} &\le c|B_\rho^+|\Big[\omega^{\frac{p_2}{p_2-1}}(2\rho)\log^{\frac{p_2}{p_2-1}}\Big(\frac{1}{\rho}\Big)\Big]^{\frac{p_2-1}{p_2}}\bigg(\fint_{B_\rho^+(x_0)} 1+|Du|^{p_2}\,dx\bigg)^{\frac{p_2-1}{p_2}} \\
&\qquad\qquad\qquad\qquad\times\bigg(\fint_{B_\rho^+(x_0)} 1+|Dv|^{p_2}+|Du|^{p_2}\,dx\bigg)^{\frac{1}{p_2}} \\
&\le c|B_\rho^+|\omega(2\rho)\log\Big(\frac{1}{2\rho}\Big)\bigg(\fint_{B_{2\rho}^+(x_0)} 1+|Du|^{p_2}\,dx\bigg)^{\frac{p_2-1}{p_2}}\bigg(\fint_{B_{2\rho}^+(x_0)} 1+|Du|^{p_2}\,dx\bigg)^{\frac{1}{p_2}} \\
&\le c\delta_p\int_{B_{2\rho}^+(x_0)} 1+|Du|^{p_2}\,dx.
\end{aligned}$$

The computations for III are similar, where we begin with **(L4)**, and Hölder's inequality with the same exponents to find

$$\begin{aligned}
\mathrm{III} &= c\int_{B_\rho^+(x_0)} \big[a(\hat{x},u,Du)-a(\hat{x},(u)_{x_0,\rho}^+,Du)\big]\cdot(Dv+Dg-Du)\,dx \\
&\le c\int_{B_\rho^+(x_0)} \big|a(\hat{x},u,Du)-a(\hat{x},(u)_{x_0,\rho}^+,Du)\big||Dv+Dg-Du|\,dx \\
&\le c\int_{B_\rho^+(x_0)} \omega_\xi\big(|u-(u)_{x_0,\rho}^+|\big)(1+|Du|)^{p_2-1}|Dv+Dg-Du|\,dx \\
&\le c|B_\rho^+|\bigg(\fint_{B_\rho^+(x_0)} \omega_\xi^{\frac{1}{\sigma}}\big(|u-(u)_{x_0,\rho}^+|\big)\,dx\bigg)^{\sigma}\bigg(\fint_{B_\rho^+(x_0)}(1+|Du|)^{p_2(1+\delta_0/4)}\,dx\bigg)^{\frac{p_2-1}{p_2}\frac{4p_2}{p_2(4+\delta_0)}} \\
&\qquad\qquad\qquad\qquad\times\bigg(\fint_{B_\rho^+(x_0)} |Dv+Dg-Du|^{p_2}\,dx\bigg)^{\frac{1}{p_2}}
\end{aligned}$$

This allows us to apply Corollary 4.2 (with $p_0 = p_2(1+\delta_0/4)$ and $p = p_2$), then use the concavity of $\omega \le 1$ with Jensen's and Poincaré's inequalities, and of course Lemma 8.5

to compute

$$
\begin{aligned}
\mathrm{III} &\le c|B_\rho^+|\bigg(\fint_{B_\rho^+(x_0)} \omega_\xi\big(|u-(u)_{x_0,\rho}^+|\big)\,dx\bigg)^{\sigma}\bigg(\fint_{B_{2\rho}^+(x_0)} 1+|Du|^{p_2}\,dx\bigg)^{\frac{p_2-1}{p_2}}\\
&\qquad\qquad\qquad\qquad\times\bigg(\fint_{B_\rho^+(x_0)} 1+|Dv|^{p_2}+|Du|^{p_2}\,dx\bigg)^{\frac{1}{p_2}}\\
&\le c|B_\rho^+|\omega_\xi^\sigma\bigg(\fint_{B_\rho^+(x_0)} \big|u-(u)_{x_0,\rho}^+\big|\,dx\bigg)\bigg(\fint_{B_{2\rho}^+(x_0)} 1+|Du|^{p_2}\,dx\bigg)^{\frac{p_2-1}{p_2}}\\
&\qquad\qquad\qquad\qquad\times\bigg(\fint_{B_{2\rho}^+(x_0)} 1+|Du|^{p_2}\,dx\bigg)^{\frac{1}{p_2}}\\
&\le c\omega_\xi^\sigma\Bigg(c\bigg(\fint_{B_\rho^+(x_0)} \rho^{p_2}+\big|u-(u)_{x_0,\rho}^+\big|^{p_2}\big)\,dx\bigg)^{\frac{1}{p_2}}\Bigg)\int_{B_{2\rho}^+(x_0)} 1+|Du|^{p_2}\,dx\\
&\le c\omega_\xi^\sigma\Bigg(\hat{c}\bigg(\rho^{p_2-n}\int_{B_\rho^+(x_0)} 1+|Du|^{p_2}\,dx\bigg)^{\frac{1}{p_2}}\Bigg)\int_{B_{2\rho}^+(x_0)} 1+|Du|^{p_2}\,dx.
\end{aligned}
$$

In estimating the final term, we recall **(G1)**. We then use Hölder's inequality and Poincaré's inequality, the fact that $u-g-v$ vanishes on the boundary, along with Lemma 8.5 to calculate

$$
\begin{aligned}
\mathrm{IV} &= c\int_{B_\rho^+(x_0)} b(x,u,Du)\cdot(u-v-g)\,dx\\
&\le c\int_{B_\rho^+(x_0)} |b(x,u,Du)||u-v-g|\,dx\\
&\le c\int_{B_\rho^+(x_0)} (1+|Du|)^{p_2-1}\,|u-v-g|\,dx\\
&\le c\bigg(\int_{B_\rho^+(x_0)}(1+|Du|)^{p_2}\,dx\bigg)^{\frac{p_2-1}{p_2}}\bigg(\int_{B_\rho^+(x_0)} |u-v-g|^{p_2}\,dx\bigg)^{\frac{1}{p_2}}\\
&\le c\bigg(\int_{B_\rho^+(x_0)} 1+|Du|^{p_2}\,dx\bigg)^{\frac{p_2-1}{p_2}}\bigg(\rho^{p_2}\int_{B_\rho^+(x_0)} |Dv+Dg-Du|^{p_2}\,dx\bigg)^{\frac{1}{p_2}}\\
&\le c\rho\bigg(\int_{B_\rho^+(x_0)} 1+|Du|^{p_2}\,dx\bigg)^{\frac{p_2-1}{p_2}}\bigg(\int_{B_\rho^+(x_0)} |Du|^{p_2}+|Dv|^{p_2}+|Dg|^{p_2}\,dx\bigg)^{\frac{1}{p_2}}\\
&\le c\rho\bigg(\int_{B_\rho^+(x_0)} 1+|Du|^{p_2}\,dx\bigg)^{\frac{p_2-1}{p_2}}\bigg(\int_{B_\rho^+(x_0)} 1+|Du|^{p_2}+|Dv|^{p_2}\,dx\bigg)^{\frac{1}{p_2}}\\
&\le c\rho\int_{B_\rho^+(x_0)} 1+|Du|^{p_2}\,dx.
\end{aligned}
$$

Combining these estimates, and writing $\hat{\omega} = \max\{\omega, \omega_\xi\}$ we find

$$
\begin{aligned}
\nu \int_{B_\rho^+(x_0)} &(1 + |Du| + |Dv + Dg|)^{p_2-2} |Du - Dv - Dg|^2 \, dx \\
&\le c\left[\hat{\omega}^\sigma\left(\hat{c}\left(\rho^{p_2-n}\int_{B_\rho^+(x_0)} 1 + |Du|^{p_2}\, dx\right)^{\frac{1}{p_2}}\right) + \rho + \delta\right] \\
&\qquad\times \int_{B_{2\rho}^+(x_0)} 1 + |Du|^{p_2}\, dx + c\delta^{1-p_2}\rho^n.
\end{aligned}
$$

Here, the use of Corollary 4.2 ensures the constants depend only on $n, N, L/\nu, \gamma_1, \gamma_2, L_0, E$, and $\|Dg\|_\infty$.

Using (3.4) from Lemma 3.7 and Lemma 8.3, we can improve this estimate to find

$$
\begin{aligned}
\int_{B_\rho^+(x_0)} 1 + |Du|^{p_2}\, dx &\le c(n, N, p_2) \int_{B_\rho^+(x_0)} (1 + |Dv + Dg|^{p_2})\, dx \\
&\quad + c\int_{B_\rho^+(x_0)} (1 + |Du|^2 + |Dv + Dg|^2)^{\frac{p_2-2}{2}} |Du - Dg - Dv|^2\, dx \\
&\le c\left(\frac{r}{\rho}\right)^\mu \int_{B_\rho^+(x_0)} (1 + |Dv + Dg|^{p_2})\, dx \\
&\quad + c\int_{B_\rho^+(x_0)} (1 + |Du|^2 + |Dv + Dg|^2)^{\frac{p_2-2}{2}} |Du - Dg - Dv|^2\, dx \\
&\le c\left[\left(\frac{r}{\rho}\right)^\mu + \hat{\omega}^\sigma\left(\hat{c}\left(\rho^{p_2-n}\int_{B_{2\rho}^+(x_0)} 1 + |Du|^{p_2}\, dx\right)^{\frac{1}{p_2}}\right) + \rho + \delta\right] \\
&\qquad\times \int_{B_{2\rho}^+(x_0)} 1 + |Du|^{p_2}\, dx + c\delta^{1-p_2}\rho^n.
\end{aligned}
$$

After rescaling ρ and keeping in mind our definition of (3.14) this yields (8.10). □

Partial regularity

We are now in a position to prove the partial regularity result Theorem 8.1 in the controllable growth case. We define

$$
\Omega_m := \left\{x \in \overline{\Omega} : p(x) > n - \frac{\delta_0}{2}\right\},
$$

where $\delta_0 > 0$ is the higher integrability exponent from Lemma 4.1. Note that whenever $n < p_2(1 + \delta_0)$, Lemma 4.1 implies that $u \in W^{1,n+\frac{\delta_0}{2}}(\Omega_m; \mathbb{R}^N)$, and so $u \in C^{1,\gamma}(\Omega_m; \mathbb{R}^N)$ where $\gamma = 1 - \frac{2n}{2n+\delta_0}$ by the Morrey-Sobolev embedding theorem (see e.g. §5.6.2 in [Eva11]).

Define the set

$$\Omega_p := \left\{ x \in \overline{\Omega} : p(x) + \frac{\delta_0}{2} \leq n < p(x) + 2 + \frac{\varepsilon_0}{2} \right\},$$

and we are now in a position to prove the following partial regularity theorem.

Theorem 8.9. *Let $u \in g + W_\Gamma^{1,p(\,\cdot\,)}(\overline{\Omega}, \mathbb{R}^N)$ be a weak solution to* (8.1) *with* (8.4) *and* (8.5) *under assumptions* **(L1)**–**(L5)**, *where the inhomogeneity satisfies the controllable growth condition* **(G1)**, *and assume $g \in C^1$. Then for $p(x)$ satisfying $n - 2 - \frac{\varepsilon_0}{2} < p(x) < n$ there holds*

$$\dim_{\mathcal{H}} \big(\mathrm{Sing}_u(\Omega_p) \big) < n - \gamma_1.$$

Moreover, we have

$$u \in C^{0,\gamma}_{loc}\big(\mathrm{Reg}_u(\Omega_p) \big),$$

for all $\gamma \in \left(0, \min\left\{1 - \frac{n-2-\frac{\varepsilon_0}{2}}{\gamma_1}, 1\right\}\right)$. Furthermore, we can characterise the singular set via the enclosure

$$\mathrm{Sing}_u(\Omega_p) \subset \Sigma_{p,\Omega_p} := \left\{ x \in \overline{\Omega} : \liminf_{\rho \downarrow 0} \rho^{p(x)-n} \mathcal{M}_p(x, r) > 0 \right\}.$$

Remark 8.10. *In fact, we can provide a local improvement of this characterisation, since by careful tracking through the calculations it is evident that for any point $x \in \mathrm{Reg}_u(\Omega_p)$ we have*

$$u \in C^{0,\hat{\gamma}}\big(N\big),$$

for all $\gamma \in \left(0, \min\left\{1 - \frac{n-2-\frac{\varepsilon_0}{2}}{p_M}, 1\right\}\right)$ and some open neighbourhood N of x. Here, of course $p_M = p_M(x) = \sup_{B_{\rho_0}(x)} p(y)$, where ρ_0 is the radius given in Corollary 4.2.

Proof of Theorem 8.9: Take κ_0 from Lemma 3.2 with $\alpha = \mu, \beta = \mu - \frac{\varepsilon_0}{2}$, $A = c$ and $B = c\delta^{1-p_2}$. The continuity of $\hat{\omega}$ lets us choose $t > 0$ small enough to ensure $c\hat{\omega}(\hat{c}t^{\frac{1}{p_2}}) < \frac{\kappa_0}{6}$. Set $\delta = \delta_p = \frac{\kappa_0}{6c}$ and fix $\tau \in (0, 1)$.

For a *regular point* $x_0 \in B^+_{1-\tau} \cup \Gamma_{1-\tau}$, i.e. $x_0 \in B^+_{1-\tau} \setminus \mathrm{Sing}_u(B^+)$, the quantity $\rho^{p(x_0)-n}\mathcal{M}(x_0, \rho)$ vanishes as $\rho \downarrow 0$, hence there exists some $0 < \hat{\rho} < \frac{\kappa_0}{6}$ with $B^+_{\hat{\rho}}(x_0) \subset\subset B^+_{1-\tau}$ and $\rho^{p(x_0)-n}\mathcal{M}(x_0, \rho) < t$ for all $0 < \rho < \hat{\rho}$. Via continuity in $y \mapsto \rho^{p(y)-n}\mathcal{M}(y, \rho)$ and further restricting $\hat{\rho}$ if necessary, we have $r^{p(y)-n}\mathcal{M}(y, r) < t$ for all $y \in B_r(x_0)$ and $0 < r < \hat{\rho}$.

Whenever $y \in \Gamma$ we proceed to calculate via (8.10), our choice of $\delta, \hat{\rho}$ and t, and

Lemma 3.2 that for $r < \hat{\rho}$

$$\begin{aligned}\mathcal{M}(y,r) &\le c\left[\left(\frac{r}{\hat{\rho}}\right)^{\mu} + \hat{\omega}^{\sigma}\left(\hat{c}\big(\hat{\rho}^{p_2-n}\mathcal{M}(y,\hat{\rho})\big)^{\frac{1}{p_2}}\right) + \hat{\rho} + 2\delta\right]\mathcal{M}(y,\hat{\rho}) + \delta^{1-p_2}\hat{\rho}^{n} \\ &\le c\left[\left(\frac{r}{\hat{\rho}}\right)^{\mu} + \frac{\kappa_0}{2}\right]\mathcal{M}(y,\hat{\rho}) + c\hat{\rho}^{\mu-\varepsilon_0} \\ &\le c\left[\left(\frac{r}{\hat{\rho}}\right)^{\mu-\varepsilon_0}\mathcal{M}(y,\hat{\rho}) + r^{\mu-\varepsilon_0}\right]. \end{aligned} \tag{8.11}$$

When $y \in B^+$, with $0 < r \le \hat{\rho} \le y_n$, the calculations are identical to those above if we instead use the interior version of Lemma 8.8.

For $y \in B^+$, with $0 < y_n < r \le \hat{\rho}$, we note that if $r > \frac{\hat{\rho}}{4}$ an estimate of the form (8.11) is trivial. On the other hand, when $r < \frac{\hat{\rho}}{4}$ we write $\hat{y} = (y_1, ..., y_{n-1}, 0)$ and have the chain of inclusions

$$B_r^+(y) \subset B_{2r}^+(\hat{y}) \subset B_{\frac{\hat{\rho}}{2}}^+(\hat{y}) \subset B_{\hat{\rho}}^+(y).$$

Consequently, again using (8.10), choice of $\delta, \delta_p, \hat{\rho}$ and t, and Lemma 3.2 that for any $r < \hat{\rho}$

$$\begin{aligned}\mathcal{M}(y,r) &\le \mathcal{M}(\hat{y},2r) \\ &\le c\left[\left(\frac{4r}{\hat{\rho}}\right)^{\mu} + \hat{\omega}^{\sigma}\left(\hat{c}\big((\hat{\rho}/2)^{p(\hat{y})-n}\mathcal{M}(\hat{y},\hat{\rho}/2)\big)^{\frac{1}{p(\hat{y})}}\right) + \frac{\hat{\rho}}{2} + 2\delta\right]\mathcal{M}(\hat{y},\hat{\rho}) + \delta^{1-p_2}\left(\frac{\hat{\rho}}{2}\right)^{n} \\ &\le c\left[\left(\frac{r}{\hat{\rho}}\right)^{\mu} + \frac{\kappa_0}{2}\right]\mathcal{M}(\hat{y},\hat{\rho}/2) + c\left(\frac{\hat{\rho}}{2}\right)^{\mu-\frac{\varepsilon_0}{2}} \\ &\le c\left[\left(\frac{r}{\hat{\rho}}\right)^{\mu-\frac{\varepsilon_0}{2}}\mathcal{M}(\hat{y},\hat{\rho}/2) + r^{\mu-\frac{\varepsilon_0}{2}}\right] \\ &\le c\left[\left(\frac{r}{\hat{\rho}}\right)^{\mu-\frac{\varepsilon_0}{2}}\mathcal{M}(y,\hat{\rho}) + r^{\mu-\frac{\varepsilon_0}{2}}\right]. \end{aligned}$$

Similarly, when $y \in B^+$, with $0 < r \le y_n \le \hat{\rho}$ we only need to consider $y_n \le \frac{\hat{\rho}}{4}$, otherwise we may simply consider the interior case. We obtain the chain of inclusions

$$B_r(y) \subset B_{y_n}(y) \subset B_{2y_n}^+(\hat{y}) \subset B_{\frac{\hat{\rho}}{2}}^+(\hat{y}) \subset B_{\hat{\rho}}^+(y),$$

and can use the interior and then boundary estimates to calculate

$$
\begin{aligned}
\mathcal{M}(y,r) &\leq c\Bigg[\left(\frac{r}{y_n}\right)^{\mu} + \hat{\omega}^{\sigma}\Big(\hat{c}\big(y_n^{p(y)-n}\mathcal{M}(y,y_n)\big)^{\frac{1}{p_2}}\Big) + y_n + 2\delta\Bigg]\mathcal{M}(y,y_n) + \delta^{1-p_2}y_n^n \\
&\leq c\left[\left(\frac{r}{y_n}\right)^{\mu} + \frac{\kappa_0}{2}\right]\mathcal{M}(y,y_n) + c y_n^{\mu-\frac{\varepsilon_0}{4}} \\
&\leq c\left[\left(\frac{r}{y_n}\right)^{\mu-\frac{\varepsilon_0}{4}}\mathcal{M}(y,y_n) + r^{\mu-\frac{\varepsilon_0}{4}}\right] \\
&\leq c\left[\left(\frac{r}{y_n}\right)^{\mu-\frac{\varepsilon_0}{4}}\mathcal{M}(\hat{y},2y_n) + r^{\mu-\frac{\varepsilon_0}{2}}\right] \\
&\leq c\left[\left(\frac{r}{y_n}\right)^{\mu-\frac{\varepsilon_0}{4}}\left[\left(\frac{4y_n}{\hat{\rho}}\right)^{\mu-\frac{\varepsilon_0}{4}}\mathcal{M}(\hat{y},\hat{\rho}/2) + \left(\frac{\hat{\rho}}{2}\right)^{\mu-\frac{\varepsilon_0}{2}}\right] + r^{\mu-\frac{\varepsilon_0}{2}}\right] \\
&\leq c\left[\left(\frac{r}{\hat{\rho}}\right)^{\mu-\frac{\varepsilon_0}{2}}\mathcal{M}(y,\hat{\rho}) + r^{\mu-\frac{\varepsilon_0}{2}}\right].
\end{aligned}
$$

This covers all cases. Restricting to the set where $p(x_0) < n < p(x_0) + 2 + \frac{\varepsilon_0}{2}$ and noting the definition of μ, ε_0, we have $n - p(x_0) < \mu - \frac{\varepsilon_0}{2} \leq n$, and so we have $Du \in L^{p,\mu-\frac{\varepsilon_0}{2}}(B_{\hat{r}}(x_0) \cap (B^+ \cup \Gamma), \mathbb{R}^{nN})$ via the definition in (2.3). By Lemma 2.1 we then have $u \in C^{0,\gamma}(B_{\hat{r}}(x_0) \cap (B^+ \cup \Gamma), \mathbb{R}^{nN})$ on this set, with $\gamma = 1 - \frac{n-\mu+\frac{\varepsilon_0}{2}}{p(x_0)}$.

The characterisation of the singular set follows from Corollary 4.2. Since $x_0 \in B^+ \cup \Gamma, \tau \in (0,1)$ were taken arbitrarily up to bounds on the exponent, and $\overline{B^+}$ is compact, we can apply a standard covering argument to conclude the result wherever these bounds hold. The dimension reduction estimate $\dim_{\mathcal{H}}(\mathrm{Sing}_u(\Omega_p) \leq n - \gamma_1$ then follows via Lemma 3.1. $\square$

Natural growth

In this section we adapt the techniques of [Ark03] and §6 of [Bec08] to deduce corresponding estimates under the *natural growth* assumption, with bounded solutions satisfying the smallness condition $2L_1\|u\|_{L^\infty} < \nu$. Note that $u \in g + W_\Gamma^{1,p(\cdot)}(B_\rho^+(x_0), \mathbb{R}^N) \cap L^\infty(B_\rho^+(x_0), \mathbb{R}^N)$, and so the admissible class of test functions must reflect this. Consequently, we now derive $L^\infty(B_{\frac{\rho}{2}}^+(x_0), \mathbb{R}^N)$ bounds for our comparison function v.

Taking some $B_r^+(y)$ with $y \in B_{\frac{\rho}{2}}^+(x_0)$ for $r < \frac{\rho}{2}$, we have via Corollary 8.4, Poincaré's inequality, Lemma 8.5 and Lemma 8.7 that

$$\begin{aligned}
\fint_{B_r^+(y)} |v|^{p_2}\,dx &\le c\left(\frac{2r}{\rho}\right)^n r^{-n}\left[\int_{B_{\frac{\rho}{2}}^+(y)} |v|^{p_2}\,dx + \rho^{p_2}\int_{B_{\frac{\rho}{2}}^+(y)} 1+|Dv|^{p_2}\,dx\right]\\
&\le c\rho^{-n}\left[\int_{B_\rho^+(x_0)} |v|^{p_2}\,dx + \rho^{p_2}\int_{B_\rho^+(x_0)} 1+|Dv|^{p_2}\,dx\right]\\
&\le c\rho^{p_2-n}\int_{B_\rho^+(x_0)} 1+|Dv|^{p_2}\,dx\\
&\le c\rho^{p_2-n}\int_{B_\rho^+(x_0)} 1+|Du|^{p_2}\,dx\\
&\le c\Big(n,N,\gamma_1,\gamma_2,\frac{L}{\nu},\frac{L_1}{\nu},\frac{L_2}{\nu},E,\omega_p,\|Dg\|_{L^\infty},\|u\|_{L^\infty}\Big).
\end{aligned}$$

Consequently,

$$\limsup_{y\in B_{\frac{\rho}{2}}^+(x_0),r\in(0,\frac{\rho}{2})} \fint_{B_r^+(y)} |v|^{p_2}\,dx \le c\Big(n,N,\gamma_1,\gamma_2,\frac{L}{\nu},\frac{L_1}{\nu},\frac{L_2}{\nu},E,\omega_p,\|Dg\|_{L^\infty},\|u\|_{L^\infty}\Big) =: m_0^p.$$

Lebesgue's Differentiation Theorem gives $v\in L^\infty(B_{\frac{\rho}{2}}^+(x_0),\mathbb{R}^N)$, with

$$\|v\|_{L^\infty(B_{\frac{\rho}{2}}^+(x_0),\mathbb{R}^N)} \le m_0.$$

The construction of an appropriate test functions becomes quite delicate, since they are required to be of a more restrictive class. However, our L^∞ bound on v allows us to estimate,

$$\begin{aligned}
\|u-g-v\|_{L^\infty(B_{\frac{\rho}{2}}^+(x_0),\mathbb{R}^N)} &\le \|u-g(x_0)\|_{L^\infty(B_{\frac{\rho}{2}}^+(x_0),\mathbb{R}^N)} + \|g(x_0)-g\|_{L^\infty(B_{\frac{\rho}{2}}^+(x_0),\mathbb{R}^N)} + m_0\\
&\le 2\|u\|_{L^\infty(B_\rho^+(x_0),\mathbb{R}^N)} + \|Dg\|_{L^\infty(B_\rho^+(x_0),\mathbb{R}^N)} + m_0 =: m, \qquad (8.12)
\end{aligned}$$

and so $u-g-v\in W_0^{1,p_2}(B_\rho^+(x_0),\mathbb{R}^N)\cap L^\infty(B_{\frac{\rho}{2}}^+(x_0),\mathbb{R}^N)$.

Fix $\delta\in(0,1)$ and set $T=T(\delta,m)=T\big(\delta,n,N,\gamma_1,\gamma_2,\frac{L}{\nu},\frac{L_1}{\nu},\frac{L_2}{\nu},E,\omega_p,\|Dg\|_{L^\infty},\|u\|_{L^\infty}\big)>0$ satisfying

$$T=2^{1+\frac{1}{\delta}}m, \qquad \text{which ensures} \qquad T^\delta-(2m)^\delta=\frac{1}{2}T^\delta.$$

Note that $T\to\infty$ as $\delta\downarrow 0$, implying

$$T^\delta-(|u-g-v|+m)^\delta \ge \frac{1}{2}T^\delta \qquad (8.13)$$

on $B^+_{\frac{\rho}{2}}(x_0)$ via (8.12), for appropriate T and δ. Furthermore,

$$h := (u-g-v)\big(T^\delta - (|u-g-v|+m)^\delta\big)_+ \in W^{1,p_2}_0\big(B^+_\rho(x_0),\mathbb{R}^N\big) \cap L^\infty\big(B^+_\rho(x_0),\mathbb{R}^N\big)$$

is an admissible test function. Here of course $(f)_+ = \max\{f,0\}$, and so h vanishes outside the set

$$\theta := \big\{x \in B^+_\rho(x_0) : |u-g-v| < T-m\big\},$$

and the weak differentiability of $u-g-v$ is inherited by h, with

$$\begin{aligned} Dh = (Du&-Dg-Dv)\big(T^\delta - (|u-g-v|+m)^\delta\big)_+ \\ &+ \delta(u-g-v) \otimes \frac{(u-g-v)\cdot(Du-Dg-Dv)}{|u-g-v|}\big(|u-g-v|+m\big)^{\delta-1}\chi(\theta), \end{aligned}$$

where $\chi(\theta)$ is the characteristic function of the set θ.

Lemma 8.11. *Let $u \in g + W^{1,p(\,\cdot\,)}_\Gamma(B^+_\rho(x_0),\mathbb{R}^N)$ be a weak solution to* (8.1) *with* (8.4) *and* (8.5) *under assumptions* **(L1)**–**(L5)***, with the inhomogeneity satisfying the natural growth condition* **(G2)** *with $2L_1\|u\|_{L^\infty} < \nu$, and $g \in C^1$. Fix $\sigma = \frac{(p_2-1)\delta_0}{p_2(4+\delta_0)}$ and $\mu := \min\{n, 2+\varepsilon_0\}$ for $\varepsilon_0 = n(1-\frac{1}{t_0}) > 0$, where t_0 is from §4.3.2 in [Bec08]. For any $x_0 \in \Gamma, \rho < 1-2\tau-|x_0|$ or $x_0 \in B^+, \rho < \min\{1-2\tau-|x_0|, (x_0)_n\}$, then there holds*

$$\begin{aligned} \mathcal{M}(x_0,r) \le c\Bigg[\left(\frac{r}{\rho}\right)^\mu &+ \hat{\omega}^\sigma\Big(\hat{c}\big(\rho^{p_2-n}\mathcal{M}(x_0,2\rho)\big)^{\frac{1}{p_2}}\Big) + \delta + \delta_p \\ &+ T^{1-\frac{p_2\delta_0}{1+\delta_0}}\big(\rho^{p_2-n}\mathcal{M}(x_0,\rho)\big)^{\frac{p_2\delta_0}{p_2(1+\delta_0)}}\Bigg]\mathcal{M}(x_0,2\rho) + \delta^{1-p_2}\rho^n, \end{aligned}$$

for each $0 < r < \rho$ and some constant c depending only on $n, N, \nu, L, \gamma_1, \gamma_2, E, \omega_p$ and $\|Dg\|$.

Proof of Lemma 8.11: The proof proceeds similarly to that of Lemma 8.8. Recalling (8.13) and Lemma 3.6, we calculate by **(L1)** and elementary integration, keeping in mind that $u-g-v$ vanishes on the boundary

$$
\begin{aligned}
\frac{1}{2}T^\delta \nu \int_{B^+_{\frac{\rho}{2}}(x_0)} & (1+|Du|+|Dv+Dg|)^{p_2-2}|Du-Dv-Dg|^2\,dx \\
&\le \nu \int_{B^+_\rho(x_0)} (1+|Du|+|Dv+Dg|)^{p_2-2}|Du-Dv-Dg|^2 \\
&\qquad\qquad \times \big(T^\delta - (|u-g-v|+m)^\delta\big)_+\,dx \\
&\le c\int_{B^+_\rho(x_0)}\int_0^1 (1+|Du+t(Dv+Dg-Du)|)^{p_2-2}|Du-Dv-Dg|^2\,dt \\
&\qquad\qquad \times \big(T^\delta - (|u-g-v|+m)^\delta\big)_+\,dx \\
&\le c\int_{B^+_\rho(x_0)}\int_0^1 D_z a_0(Du+t(Dv+Dg-Du))(Dv+Dg-Du)\cdot(Dv+Dg-Du)\,dt \\
&\qquad\qquad \times \big(T^\delta - (|u-g-v|+m)^\delta\big)_+\,dx \\
&= c\int_{B^+_\rho(x_0)} \big[a_0(Dv+Dg)-a_0(Du)\big]\cdot(Dv+Dg-Du)\big(T^\delta - (|u-g-v|+m)^\delta\big)_+\,dx \\
&= c\int_{B^+_\rho(x_0)} \big[a_0(Dv+Dg)-a_0(Dv)\big]\cdot Dh\,dx \\
&\quad + c\int_{B^+_\rho(x_0)} \big[a(x,u,Du)-a(\hat{x},u,Du)\big]\cdot Dh\,dx \\
&\quad + c\int_{B^+_\rho(x_0)} \big[a(\hat{x},u,Du)-a(\hat{x},(u)^+_{x_0,\rho},Du)\big]\cdot Dh\,dx \\
&\quad + c\int_{B^+_\rho(x_0)} b(x,u,Du)\cdot(u-g-v)\big(T^\delta - (|u-g-v|+m)^\delta\big)_+\,dx \\
&\quad + c\int_{B^+_\rho(x_0)} \big[a_0(Dv+Dg)-a_0(Du)\big]\cdot\delta(u-g-v) \\
&\qquad\qquad \otimes \frac{(u-g-v)\,\cdot\,(Du-Dg-Dv)}{|u-g-v|}\big(|u-g-v|+m\big)^{\delta-1}\chi(\theta)\,dx \\
&= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.
\end{aligned}
$$

Since $\big(T^\delta-(|u-g-v|+m)^\delta\big)_+ \le T^\delta$, we can use the same estimates as in Lemma 8.8 to deduce

$$
\begin{aligned}
\mathrm{I}+\mathrm{II}+\mathrm{III} \le cT^\delta\Bigg[\hat{\omega}^\sigma\Bigg(\hat{c}\bigg(\rho^{p_2-n}\int_{B^+_\rho(x_0)} 1+|Du|^{p_2}\,dx\bigg)^{\frac{1}{p_2}}\Bigg)+\delta+\delta_p\Bigg]\int_{B^+_{2\rho}(x_0)} 1+|Du|^{p_2}\,dx \\
+\delta^{1-p_2}\rho^n.
\end{aligned}
$$

with the constant c only depending on $n, N, \gamma_1, \gamma_2, L/\nu, E, \omega_p$ and $\|Dg\|_{L^\infty}$.

In estimating the inhomogeneity we can use **(G2)**, Hölder's inequality then Corollary 4.2, the estimate $|u-g-v|\chi(\theta) < T-m \le T$, Poincaré's inequality and Lemma 8.5

to calculate

$$
\begin{aligned}
\mathrm{IV} &\le c\int_{B_\rho^+(x_0)} |b(x,u,Du)||u-g-v|\big(T^\delta-(|u-g-v|+m)^\delta\big)_+\,dx\\
&\le cT^\delta\int_{B_\rho^+(x_0)}\big(L_1|Du|^{p(x)}+L_2\big)|u-g-v|\chi(\theta)\,dx\\
&\le cT^\delta|B_\rho^+(x_0)|\fint_{B_\rho^+(x_0)}(1+|Du|^{p_2})|u-g-v|\chi(\theta)\,dx\\
&\le cT^\delta|B_\rho^+(x_0)|\bigg(\fint_{B_\rho^+(x_0)}(1+|Du|^{p_2})^{1+\delta_0}\,dx\bigg)^{\frac{1}{1+\delta_0}}\bigg(\fint_{B_\rho^+(x_0)}|u-g-v|\chi(\theta)^{\frac{1+\delta_0}{\delta_0}}\,dx\bigg)^{\frac{\delta_0}{1+\delta_0}}\\
&\le cT^\delta|B_\rho^+(x_0)|\bigg(\fint_{B_{2\rho}^+(x_0)}1+|Du|^{p_2}\,dx\bigg)\bigg(\fint_{B_\rho^+(x_0)}|u-g-v|\chi(\theta)^{\frac{1+\delta_0}{\delta_0}-p_2+p_2}\,dx\bigg)^{\frac{\delta_0}{1+\delta_0}}\\
&\le cT^\delta\bigg(\int_{B_{2\rho}^+(x_0)}1+|Du|^{p_2}\,dx\bigg)\bigg(\fint_{B_\rho^+(x_0)}T^{\frac{1+\delta_0}{\delta_0}-p_2}|u-g-v|^{p_2}\,dx\bigg)^{\frac{\delta_0}{1+\delta_0}}\\
&\le cT^{\delta+1-\frac{p_2\delta_0}{1+\delta_0}}\bigg(\int_{B_{2\rho}^+(x_0)}1+|Du|^{p_2}\,dx\bigg)\bigg(\fint_{B_\rho^+(x_0)}\rho^{p_2}|Du-Dg-Dv|^{p_2}\,dx\bigg)^{\frac{\delta_0}{1+\delta_0}}\\
&\le cT^{\delta+1-\frac{p_2\delta_0}{1+\delta_0}}\bigg(\int_{B_{2\rho}^+(x_0)}1+|Du|^{p_2}\,dx\bigg)\bigg(\rho^{p_2}\fint_{B_\rho^+(x_0)}|Du-Dg|^{p_2}+|Dv|^{p_2}\,dx\bigg)^{\frac{\delta_0}{1+\delta_0}}\\
&\le cT^{\delta+1-\frac{p_2\delta_0}{1+\delta_0}}\bigg(\rho^{p_2-n}\int_{B_\rho^+(x_0)}1+|Du|^{p_2}\,dx\bigg)^{\frac{\delta_0}{1+\delta_0}}\int_{B_{2\rho}^+(x_0)}1+|Du|^{p_2}\,dx.
\end{aligned}
$$

Here we have restricted $1+\delta_0<\frac{p_2}{p_2-1}$ if necessary, to ensure $1-\frac{p_2\delta_0}{1+\delta_0}>0$, and the constant c now also depends on L_1/ν and L_2.

In estimating the final term, we use **(L2)** and Young's inequality and (8.13), then

Young's inequality and Lemma 8.5 to see

$$
\begin{aligned}
\mathrm{V} &\le c\delta \int_{B_\rho^+(x_0)} \big|a_0(Dv+Dg)-a_0(Du)\big||u-g-v| \\
&\qquad\qquad \times \frac{|u-g-v||Du-Dg-Dv|}{|u-g-v|}\left(|u-g-v|+m\right)^{\delta-1}\chi(\theta)\,dx \\
&\le c\delta \int_{B_\rho^+(x_0)} \big[(1+|Dv+Dg|)^{p_2-1}+(1+|Du|)^{p_2-1}\big]|u-g-v| \\
&\qquad\qquad \times |Du-Dg-Dv|\big(|u-g-v|+m\big)^{\delta-1}\chi(\theta)\,dx \\
&\le cT^\delta\delta \int_{B_\rho^+(x_0)} \big(1+|Dv+Dg|^{p_2-1}+|Du|^{p_2-1}\big)|Du-Dg-Dv|\,dx \\
&\le cT^\delta\delta \int_{B_\rho^+(x_0)} 1+|Dv|^{p_2}+|Du|^{p_2}\,dx \\
&\le cT^\delta\delta \int_{B_\rho^+(x_0)} 1+|Du|^{p_2}\,dx.
\end{aligned}
$$

Combining these estimates, we recall our definition of $\mathcal{M}(x,r)$ from (3.14) to find

$$
\begin{aligned}
&\nu \int_{B_{\frac{\rho}{2}}^+(x_0)} (1+|Du|+|Dv+Dg|)^{p_2-2}|Du-Dv-Dg|^2\,dx \qquad (8.14)\\
&\le c\bigg[\hat{\omega}^\sigma\Big(\hat{c}\big(\rho^{p_2-n}\mathcal{M}(x_0,\rho)\big)^{\frac{1}{p_2}}\Big)+\delta+T^{1-\frac{p_2\delta_0}{1+\delta_0}}\big(\rho^{p_2-n}\mathcal{M}(x_0,\rho)\big)^{\frac{\delta_0}{1+\delta_0}}\bigg]\mathcal{M}(x_0,2\rho)+\delta^{1-p_2}\rho^n.
\end{aligned}
$$

Lemma 3.7 gives us

$$
1+|Du|^{p_2} \le c\Big[(1+|Dv+Dg|^{p_2})+(1+|Du|+|Dv+Dg|)^{p_2-2}|Du-Dg-Dv|^2\Big]
$$

and using Lemma 8.3, we can improve (8.14) to find for $r<\rho$

$$
\begin{aligned}
\mathcal{M}(x_0,r) &\le c(n,N,p_2)\int_{B_r^+(x_0)} (1+|Dv+Dg|^{p_2})\,dx \\
&\qquad + c\int_{B_r^+(x_0)} (1+|Du|^2+|Dv+Dg|^2)^{\frac{p_2-2}{2}}|Du-Dg-Dv|^2\,dx \\
&\le c\left(\frac{r}{\rho}\right)^\mu \int_{B_\rho^+(x_0)} (1+|Dv+Dg|^{p_2})\,dx \\
&\qquad + c\int_{B_\rho^+(x_0)} (1+|Du|^2+|Dv+Dg|^2)^{\frac{p_2-2}{2}}|Du-Dg-Dv|^2\,dx \\
&\le c\bigg[\left(\frac{r}{\rho}\right)^\mu+\hat{\omega}^\sigma\Big(\hat{c}\big(\rho^{p_2-n}\mathcal{M}(x_0,2\rho)\big)^{\frac{1}{p_2}}\Big)+\delta \\
&\qquad + T^{1-\frac{p_2(s-1)}{s}}\Big(\rho^{p_2-n}\mathcal{M}(x_0,\rho)\Big)^{\frac{p_2\delta_0}{p_2(1+\delta_0)}}\bigg]\mathcal{M}(x_0,2\rho)+\delta^{1-p_2}\rho^n,
\end{aligned}
$$

as required. $\square$

Partial regularity

We are now in a position to prove the partial regularity result corresponding to Theorem 8.9, this time under the natural growth condition.

Theorem 8.12. *Let $u \in g + W_{\Gamma}^{1,p(\,\cdot\,)}(B_\rho^+(x_0), \mathbb{R}^N)$ be a weak solution to* (8.1) *with* (8.4) *and* (8.5) *under assumptions* **(L1)–(L5)***, with the inhomogeneity satisfying the natural growth condition* **(G2)***, with $2L_1\|u\|_{L^\infty} < \nu$ and $g \in C^1$. Then for $p(x) < n < p(x)+2+\frac{\varepsilon_0}{2}$ there holds*

$$\dim_{\mathcal{H}}\big(\operatorname{Sing}_u(\Omega_p)\big) < n - \gamma_1.$$

Moreover, we have

$$u \in C^{0,\gamma}_{loc}\big(\operatorname{Reg}_u(\Omega_p)\big),$$

for all $\gamma \in \left(0, \min\left\{1 - \frac{n-2-\frac{\varepsilon_0}{2}}{\gamma_1}, 1\right\}\right)$. Furthermore, we can characterise the singular via the enclosure

$$\operatorname{Sing}_u(\Omega_p) \subset \Sigma_{p,\Omega_p} := \left\{x \in \overline{\Omega} : \liminf_{\rho\downarrow 0} \rho^{p(x)-n}\mathcal{M}_p(x,u,r) > 0\right\}.$$

Remark 8.13. *In fact, we can provide a local improvement of this characterisation, since by careful tracking through the calculations it is evident that for any point $x \in \operatorname{Reg}_u(\Omega_p)$ we have*

$$u \in C^{0,\hat{\gamma}}\big(N\big),$$

for all $\gamma \in \left(0, \min\left\{1 - \frac{n-2-\frac{\varepsilon_0}{2}}{p_M}, 1\right\}\right)$ and some open neighbourhood N of x. Here, of course $p_M = p_M(x) = \sup_{B_{\rho_0}(x)} p(y)$, where ρ_0 is the radius given in Corollary 4.2.

Proof of Theorem 8.12: Take κ_0 from Lemma 3.2 with $\alpha = \mu, \beta = \mu - \frac{\varepsilon_0}{2}$, and set $\delta = \delta_p = \frac{\kappa_0}{6}$ and $\tau \in (0, \frac{1}{2})$. This fixes $T > 0$. Noting the continuity of $\hat{\omega}$, we can fix then $t > 0$ small enough to ensure

$$\min\left\{\hat{\omega}(\hat{c}t^{\frac{1}{p_2}}), T^{1-\frac{p_2(s-1)}{s}} t^{\frac{(s-1)p_2}{sp_2}}\right\} < \frac{\kappa_0}{6c}.$$

Choosing a regular point $x_0 \in B_{1-\tau}^+ \cup \Gamma$ we recall that $\rho^{p(x_0)-n}\mathcal{M}(x_0,\rho)$ vanishes as $\rho \downarrow 0$, and so choose $0 < \hat{\rho} < \frac{\kappa_0}{3}$ with $B_{\hat{\rho}}^+(x_0) \subset\subset B_{1-\tau}^+$ and $\hat{\rho}^{p(x_0)-n}\mathcal{M}(x_0,\hat{\rho}) < t$. Since $y \mapsto \hat{\rho}^{p(y)-n}\mathcal{M}(y,\hat{\rho})$ is continuous, after restricting $\hat{\rho}$ further if necessary, we have $\hat{\rho}^{p(y)-n}\mathcal{M}(y,\hat{\rho}) < t$ for all $y \in B_{\hat{r}}(x_0)$.

The remainder of the proof is now completely analogous to Theorem 8.9, and we refer the reader to its proof for details. $\square$

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